

Hardy's test of quantum mechanics

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<http://www.oberlin.edu/physics/dstyer/StrangeQM/Hardy.pdf>

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Hardy's experiment is in many ways the cleanest test of quantum mechanics ever performed, because it is a sensitive probe of the most vulnerable aspects of the theory. I don't give the experiment a lot of attention in my book *The Strange World of Quantum Mechanics* (Cambridge University Press, Cambridge, U.K., 2000), in part because it hadn't yet been performed when the book was written. This document supplements that book, and it assumes you've read chapters 1 through 4 and pages 38 through 40 (chapters 6 and 7 would also be beneficial).

In Hardy's experiment a particular outcome never occurs according to local determinism and sometimes occurs according to quantum mechanics. The experiment, a variation on the Einstein–Podolsky–Rosen idea, was proposed by Lucien Hardy in 1993 and performed by Paul Kwiat and his coworkers at the Los Alamos National Laboratory in early 1999. The results were a thrilling confirmation of quantum mechanics: the outcome that would never happen under local determinism did in fact sometimes occur, and the measured probability of occurrence was exactly the probability predicted by quantum theory.

1 The apparatus

A source rests between two detectors, one to the left and one to the right. The source ejects two atoms, one toward each detector, in an unusual entangled state. Each detector is a Stern–Gerlach analyzer that can be set to either of two different orientations: vertical or tilted by 76.35° . The two possible orientations on the left are called L1 and L2; those on the right are called R1 and R2.

I wish I were able to tell you about this unusual state. The state is easy to describe in the mathematical language of quantum theory and it is not hard to produce in the laboratory, but I don't know how to describe it at all using every-day language. Similarly, I will not be able to tell you why the magic angle is 76.35° , although I can tell you that almost any angle would work ... this angle just happens to result in the largest quantal probability and hence the greatest deviation from local determinism. These questions are answered in a technical way in the “technical appendix” to this document (section 5).

2 The prediction of quantum mechanics

Quantum mechanics can predict the probability of any situation: For example, if the orientations are set to L1 on the left and R2 on the right, what is the probability that the left atom comes out + and the right atom comes out −? There are sixteen such predictions to make, but we are interested in only four of them:

Experiment A		
orientations:	L1	R1
results:	−	−
happens:	never	

Experiment B		
orientations:	L1	R2
results:	+	+
happens:	never	

Experiment C		
orientations:	L2	R1
results:	+	+
happens:	never	

Experiment D		
orientations:	L2	R2
results:	+	+
happens:	sometimes	

In experiment D, the result “both atoms come out +” happens with a probability of 9.017%. I can’t justify these predictions without a technical argument (see the appendix) but I can tell you that experiment solidly confirms them. (One advantage of the Einstein–Podolsky–Rosen argument given in chapter 6 of *The Strange World of Quantum Mechanics* is that the quantal prediction there is straightforward.)

Aside: One thing that intrigues me about Hardy’s test is the mathematical origin of the probability 0.09017... The number is g^5 , where the constant g is equal to $(\sqrt{5} - 1)/2 = 0.6180\dots$ and is called “the golden mean”. If a line of length 1 is divided into two pieces so that the ratio of the length of the whole to the length of the long piece is equal to the ratio of the length of the long piece to the length of the short piece, then the long piece will have length g . The ancient Greeks considered a rectangle of width 1 and height g to be the “ideal” (most beautiful) rectangle. The Parthenon in Athens, for example, has a height of g times its width. Why do we write on $8\frac{1}{2}$ by 11 inch paper instead of, say, 10 by 10 inch squares? Because a rectangle with aspect ratio $8\frac{1}{2}/11 = 0.7727$ — nearly a golden rectangle — is considered more beautiful than a square. The golden rectangle comes up in the work of Leonardo da Vinci, Titian, and Mondrian. In addition it is connected with the Great Pyramid, the star pentagram (which in one form appears in the American flag and which in another is said to call up the devil), the Fibonacci sequence, recursion relations, and with algorithms for locating the minimum of a one-variable function. But this is the first time I’ve ever seen it appear in quantum mechanics.

3 The prediction of local determinism

In any local deterministic scheme, the atoms are launched towards their detectors with instructions sets, such as: “If the detector is set to 1, come out +. If it is set to 2, come out −.” I will represent this instruction set by the symbol

$$\begin{pmatrix} 1:+ \\ 2:- \end{pmatrix}.$$

My goal is to list all the instruction sets consistent with the predictions for all four experiments above. First, suppose that the atom going left has instructions to come out + if the experiment it encounters is L2. Then the possible instruction sets are symbolized by:

$$\leftarrow \begin{pmatrix} \text{L1: ?} \\ \text{L2: +} \end{pmatrix} \qquad \begin{pmatrix} \text{R1: ?} \\ \text{R2: ?} \end{pmatrix} \longrightarrow$$

According to experiment C, if L2 gives +, then R1 can't give + so it must give -. The possible instruction sets are then:

$$\leftarrow \begin{pmatrix} \text{L1: ?} \\ \text{L2: +} \end{pmatrix} \qquad \begin{pmatrix} \text{R1: -} \\ \text{R2: ?} \end{pmatrix} \longrightarrow$$

According to experiment A, if R1 gives - then L1 must give +:

$$\leftarrow \begin{pmatrix} \text{L1: +} \\ \text{L2: +} \end{pmatrix} \qquad \begin{pmatrix} \text{R1: -} \\ \text{R2: ?} \end{pmatrix} \longrightarrow$$

And according to experiment B, if L1 gives + then R2 must give -:

$$\leftarrow \begin{pmatrix} \text{L1: +} \\ \text{L2: +} \end{pmatrix} \qquad \begin{pmatrix} \text{R1: -} \\ \text{R2: -} \end{pmatrix} \longrightarrow$$

This instruction set is the only one consistent with predictions A, B, and C and with L2 giving the result +. And this instruction set shows that in situation D (for which the detectors are set to L2 and R2) whenever the left atom comes out +, then the right atom *has to* come out -. But according to quantum mechanics (experiment D), in this case the atom on the right *might come out either - or +*. There are *no* instruction sets consistent with all the predictions of quantum mechanics.

No local deterministic theory — no matter how clever, or intuitive, or appealing — can predict the results that occur in nature. Quantum mechanics can.

4 Locality

Look at experiment B. If the orientations are L1 and R2, and if the result at R2 is +, then the result at L1 must be -. From experiment A (concerning orientations L1 and R1), we see that if the result at L1 is -, then the result at R1 must be +. We conclude:

Statement I: Suppose L1 is measured on the left and R2 is measured on the right giving +. Then if R1 had instead been measured on the right, the result would have been +.

Now look at experiment D. If the orientations are L2 and R2, and if the result at R2 is +, then the result at L2 might be either + or -. Suppose it is +. Then experiment C tells us that if R1 had instead been measured, the result would have been -. In conclusion:

Statement II: Suppose L2 is measured on the left and R2 is measured on the right giving +. Then if R1 had instead been measured on the right, the result might have been –.

Statements I and II are both correct. They show that by changing the experiment performed on the left (from L1 to L2), you can change something about the right (from “would have been +” to “might have been –”). Is this a nonlocal way to instantly influence the right from the left?

No. When you change the orientation of the left detector, you don’t change what happens on the right. Instead you change what would have happened on the right if you had performed an experiment other than the one which you actually did perform. (You could have performed it, but in fact you did not.) The second sentences of statements I and II are called “counterfactual statements”: they concern experiments that, in fact, have not been performed. The change in the counterfactual statement is not an “effect” in the usual sense of the phrase “cause and effect”. You cannot send a message by changing the truth or falsity of a counterfactual statement.

Summary: An experiment performed at one place can instantly change the wavefunction at a distant place. The decision of which experiment to perform at one place can instantly change the interpretation of distant experiments that are not performed. But the decision of which experiment to perform at one place *cannot* instantly change the results of distant experiments that *are* performed. Quantum mechanics exhibits correlation without causality, and the word “local” *means* local causality. Quantum mechanics takes us to the brink of implausibility — *but not beyond*.

5 Technical appendix

This appendix uses the full mathematical machinery of quantum theory to describe Hardy’s “unusual entangled state” and to uncover the “prediction of quantum mechanics” given in section 2. Read it only if you’re familiar with terms like “ket”, “inner product”, and “Dirac notation”. If you are familiar with those terms, then this appendix will show you how the mathematics illuminates the verbal arguments and vice versa. If you aren’t familiar with those terms, then you will find this appendix to be absolutely opaque.

The situation

Recall that the apparatus demands a Stern-Gerlach analyzer on the left, with two possible orientations L1 and L2, and a Stern-Gerlach analyzer on the right, with two possible orientations R1 and R2. In the body of this document we took orientation #1 to be vertical and orientation #2 to be tilted by the same amount both on the left and the right. This appendix does not make those assumptions: instead L1 and R1 are arbitrary orientations, while L2 is rotated from L1 by the angle θ and R2 is rotated from R1 by the angle ϕ . It might happen L1 and L2 are vertical, and it might happen that $\theta = \phi$, but this is not required.

In this appendix, one-body states are represented by notation like $|L1:+\rangle$, meaning that if the left-bound atom encounters its detector in orientation #1, it will (with certainty) leave through the + exit. Two-body

states are represented by notation like $|L2:+, R1:-\rangle$, meaning that if the left-bound atom encounters the left detector in orientation #2, it will (with certainty) leave through the + exit, and that if the right-bound atom encounters the right detector in orientation #1, it will (with certainty) leave through the – exit.

Of course, there are one-body states for left-bound atoms other than $|L1:+\rangle$ (+ with certainty) and $|L1:-\rangle$ (– with certainty); these are represented by the superposition

$$a |L1:+\rangle + b |L1:-\rangle,$$

where a and b are complex numbers. If a left-bound atom in this state enters the left analyzer set to orientation #1, then it has probability $|a|^2$ of emerging from the + exit and probability $|b|^2$ of emerging from the – exit (whence $|a|^2 + |b|^2 = 1$).

Similarly, the most general two-body state is the superposition

$$|\psi\rangle = a |L1:+, R1:+\rangle + b |L1:+, R1:-\rangle + c |L1:-, R1:+\rangle + d |L1:-, R1:-\rangle \quad (1)$$

where

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1.$$

To the well-trained quantum mechanic, writing down this superposition is as easy and as natural as rolling over in bed. It is so natural that the well-trained mechanic might not realized that we have already taken the essential step that is so deadly to our classical intuition: we have written down an entangled state. If the pair of atoms is in a product state such as

$$|L1:-, R1:+\rangle = |L1:-\rangle |R1:+\rangle,$$

then observing the left atom (with any detector orientation) has no bearing on the right atom; it remains in state $|R1:+\rangle$. Similarly for the most general product state

$$(\alpha |L1:+\rangle + \beta |L1:-\rangle)(\gamma |R1:+\rangle + \delta |R1:-\rangle).$$

But in general, this independence between left and right does *not* hold for the arbitrary superposition $|\psi\rangle$ in equation (1).

The entangled state

Our task is to find coefficients a , b , c , and d in equation (1) which give rise to the four outcomes described in “the prediction of quantum mechanics”. First, to ensure the result of experiment A, we select $d = 0$ whence

$$|\psi\rangle = a |L1:+, R1:+\rangle + b |L1:+, R1:-\rangle + c |L1:-, R1:+\rangle.$$

We go on to experiment B. The amplitude to find state $|\psi\rangle$ in state $|L1:+, R2:+\rangle$ is of course $\langle L1:+, R2:+ | \psi \rangle$. To ensure the result of experiment B, this amplitude must vanish:

$$\begin{aligned} 0 &= \langle L1:+, R2:+ | \psi \rangle \\ &= a \langle L1:+, R2:+ | L1:+, R1:+ \rangle + b \langle L1:+, R2:+ | L1:+, R1:- \rangle + c \langle L1:+, R2:+ | L1:-, R1:+ \rangle \\ &= a \langle R2:+ | R1:+ \rangle + b \langle R2:+ | R1:- \rangle + c (0) \end{aligned}$$

In conclusion,

$$b \langle R2: + | R1:- \rangle = -a \langle R2: + | R1:+ \rangle. \quad (2)$$

Similarly for experiment C:

$$\begin{aligned} 0 &= \langle L2:+, R1:+ | \psi \rangle \\ &= a \langle L2:+, R1:+ | L1:+, R1:+ \rangle + b \langle L2:+, R1:+ | L1:+, R1:- \rangle + c \langle L2:+, R1:+ | L1:-, R1:+ \rangle \\ &= a \langle L2: + | L1:+ \rangle + b(0) + c \langle L2: + | L1:- \rangle \end{aligned}$$

whence

$$c \langle L2: + | L1:- \rangle = -a \langle L2: + | L1:+ \rangle. \quad (3)$$

The resulting quantal probability

If the state $|\psi\rangle$ satisfies the two restrictions embodied in equations (2) and (3) above, then it gives the desired results for experiments A, B, and C. If such a state is subjected to experiment D, what is the probability p of the outcome “both atoms exit through +”?

The amplitude for this outcome is

$$\begin{aligned} &\langle L2:+, R2: + | \psi \rangle \\ &= a \langle L2:+, R2: + | L1:+, R1:+ \rangle + b \langle L2:+, R2: + | L1:+, R1:- \rangle + c \langle L2:+, R2: + | L1:-, R1:+ \rangle \\ &= a \langle L2: + | L1:+ \rangle \langle R2: + | R1:+ \rangle + b \langle L2: + | L1:+ \rangle \langle R2: + | R1:- \rangle + c \langle L2: + | L1:- \rangle \langle R2: + | R1:+ \rangle. \end{aligned}$$

Using results (2) and (3), this amplitude becomes

$$\begin{aligned} &\langle L2:+, R2: + | \psi \rangle \\ &= a \langle L2: + | L1:+ \rangle \langle R2: + | R1:+ \rangle - a \langle L2: + | L1:+ \rangle \langle R2: + | R1:+ \rangle - a \langle L2: + | L1:+ \rangle \langle R2: + | R1:+ \rangle \\ &= a \langle L2: + | L1:+ \rangle \langle R2: + | R1:+ \rangle. \end{aligned}$$

The probability is this amplitude squared, so

$$p = |a|^2 |\langle L2: + | L1:+ \rangle|^2 |\langle R2: + | R1:+ \rangle|^2.$$

The two inner products on the right are familiar from page 26 of *The Strange World of Quantum Mechanics*: $|\langle R2: + | R1:+ \rangle|^2$ is the probability that an atom, with spin up relative to orientation #1, comes out of the + exit of a Stern-Gerlach analyzer in orientation #2. We define

$$p_r \equiv |\langle R2: + | R1:+ \rangle|^2 = \cos^2(\phi/2), \quad (4)$$

$$p_\ell \equiv |\langle L2: + | L1:+ \rangle|^2 = \cos^2(\theta/2),$$

so

$$p = |a|^2 p_r p_\ell. \quad (5)$$

This is the probability that, if experiment D is performed, both atoms come out +. In any local deterministic theory, this number would be zero. In quantum mechanics it is clearly non-zero except for certain special values of the angles, and experiment confirms quantum mechanics. So we've already gone as far as we need to go. However, there's a cute result that comes from asking what happens if we look for the experiment that gives the largest possible value for p , in other words the largest possible deviation from the local determinist prediction of zero.

The maximum resulting quantal probability

It appears, from equation (5), that the probability p depends upon the experimental setup through p_r and p_ℓ (which depend upon the detector angles θ and ϕ), and upon the initial state through $|a|^2$. Surprisingly, the quantity $|a|^2$ is fixed by the experimental setup as well. To show this, take the squares of equations (2) and (3) to find

$$\begin{aligned} |b|^2(1 - p_r) &= |a|^2 p_r \\ |c|^2(1 - p_\ell) &= |a|^2 p_\ell. \end{aligned}$$

The normalization condition is

$$\begin{aligned} 1 &= |a|^2 + |b|^2 + |c|^2 \\ &= |a|^2 + |a|^2 \frac{p_r}{1 - p_r} + |a|^2 \frac{p_\ell}{1 - p_\ell} \\ &= |a|^2 \left(\frac{(1 - p_r)(1 - p_\ell) + p_r(1 - p_\ell) + p_\ell(1 - p_r)}{(1 - p_r)(1 - p_\ell)} \right) \\ &= |a|^2 \left(\frac{1 - p_r p_\ell}{(1 - p_r)(1 - p_\ell)} \right) \end{aligned} \tag{6}$$

whence $|a|^2$ is determined and equation (5) becomes

$$p = \frac{p_r(1 - p_r)p_\ell(1 - p_\ell)}{1 - p_r p_\ell}. \tag{7}$$

Note that this result is symmetric under the interchange of p_r and p_ℓ .

Our task now is to find the maximum value of p given that the variables p_r and p_ℓ are restricted to the unit square ($0 \leq p_r \leq 1$, $0 \leq p_\ell \leq 1$). The maximum lies either on the boundary of the square or at an interior point where

$$\frac{\partial p}{\partial p_r} = 0 \quad \text{and} \quad \frac{\partial p}{\partial p_\ell} = 0.$$

At all points on the boundary, $p = 0$, so the maximum doesn't fall on the boundary. To locate the interior point, take the derivative

$$\frac{\partial p}{\partial p_r} = \frac{[(1 - p_r p_\ell)(1 - 2p_r) - p_r(1 - p_r)(-p_\ell)]}{(1 - p_r p_\ell)^2} p_\ell(1 - p_\ell) = \frac{[1 - 2p_r + p_r^2 p_\ell]}{(1 - p_r p_\ell)^2} p_\ell(1 - p_\ell)$$

whence the maximum is located where

$$0 = 1 - 2p_r + p_r^2 p_\ell. \tag{8}$$

Because equation (7) is symmetric under interchange, the corresponding equation derived from the derivative with respect to p_ℓ is

$$0 = 1 - 2p_\ell + p_\ell^2 p_r. \quad (9)$$

The difference between equations (8) and (9) is

$$0 = -2(p_r - p_\ell) + p_r p_\ell (p_r - p_\ell) = (p_r p_\ell - 2)(p_r - p_\ell),$$

whence the maximum value of p occurs when

$$p_r = p_\ell.$$

Substituting this requirement back into equation (8) (or into equation (9)) gives

$$0 = 1 - 2p_r + p_r^3 = (p_r - 1)(p_r^2 + p_r - 1)$$

This cubic equation has three roots:

$$p_r = 1, \frac{\pm\sqrt{5} - 1}{2}.$$

We have already seen that $p_r = p_\ell = 1$ gives rise to $p = 0$, which is not a maximum. And the point $p_r = p_\ell = (-\sqrt{5} - 1)/2$ gives an impossible negative value for the probabilities p_r and p_ℓ . Thus the maximum value for the quantal probability p occurs when

$$p_r = p_\ell = \frac{\sqrt{5} - 1}{2}. \quad (10)$$

As mentioned earlier, this value is called the golden mean

$$g \equiv \frac{\sqrt{5} - 1}{2} = 0.6180\dots$$

Substituting this value for p_r and p_ℓ into equation (7) shows that the maximum value of p is

$$p = g^5 = 0.0901\dots$$

This value is achieved when the experimental arrangement has angle ϕ given by

$$p_r = \cos^2(\phi/2) = g, \quad \text{whence} \quad \phi = 76.35^\circ$$

and similarly for angle θ . We can use equation (6) to find the coefficient a , and equations (2) and (3) to find b and c (using the phase convention that $\langle \text{L2:} + | \text{L1:} + \rangle = \cos(\theta/2)$ and $\langle \text{L2:} + | \text{L1:} - \rangle = \sin(\theta/2)$). This fixes the state giving rise to the maximum quantal probability as

$$|\psi\rangle = g [\sqrt{g} | \text{L1:} +, \text{R1:} + \rangle - | \text{L1:} +, \text{R1:} - \rangle - | \text{L1:} -, \text{R1:} + \rangle]. \quad (11)$$

Quantal states for situations with symmetric apparatus

Suppose the apparatus at the left and right ends of the experiment are the same, that is, experiment 1 has the same orientation on both sides and experiment 2 has the same orientation on both sides. (This is the case for the apparatus described in section 1, “The apparatus”, in this document.) In this situation the labels “L” and “R” can sometimes be dropped. For example

$$\langle R2: + | R1:- \rangle = \langle L2: + | L1:- \rangle \equiv \langle 2: + | 1:- \rangle,$$

and, of course, from definition (4)

$$p_r = p_\ell = |\langle 2: + | 1:+ \rangle|^2.$$

Notice that in this situation equations (2) and (3) are identical except that (2) involves “R” and (3) involves “L”. In this symmetric situation, therefore,

$$b = c = -a \frac{\langle 2: + | 1:+ \rangle}{\langle 2: + | 1:- \rangle}.$$

It follows that the superposition state (1) is

$$|\psi\rangle = a \left[|L1:+, R1:+\rangle - \frac{\langle 2: + | 1:+ \rangle}{\langle 2: + | 1:- \rangle} (|L1:+, R1:-\rangle + |L1:-, R1:+\rangle) \right].$$

Meanwhile, equation (6) becomes

$$|a|^2 = \frac{1 - p_r}{1 + p_r}$$

and (4) becomes

$$|\langle 2: + | 1:+ \rangle|^2 = p_r.$$

But $|\langle 2: + | 1:+ \rangle|^2 + |\langle 2: + | 1:- \rangle|^2 = 1$ so

$$|\langle 2: + | 1:- \rangle|^2 = 1 - p_r.$$

Taking square roots of the previous three expressions gives (with the appropriate choice of phase)

$$\begin{aligned} a &= \sqrt{\frac{1 - p_r}{1 + p_r}}, \\ \langle 2: + | 1:+ \rangle &= \sqrt{p_r}, \\ \langle 2: + | 1:- \rangle &= \sqrt{1 - p_r}. \end{aligned}$$

Pulling everything together, the entangled superposition state is

$$|\psi\rangle = \sqrt{\frac{1 - p_r}{1 + p_r}} |L1:+, R1:+\rangle - \sqrt{\frac{p_r}{1 + p_r}} (|L1:+, R1:-\rangle + |L1:-, R1:+\rangle),$$

for which the probability associated with experiment D is (from 7)

$$p = p_r^2 \frac{1 - p_r}{1 + p_r}.$$

You can see that as p_r varies from 0 to 1 the state $|\psi\rangle$ varies from non-entangled to entangled to “maximally entangled”, while p varies from 0 to nonzero to 0 again. (The maximum occurs when $p_r = g$.) In their article “The mystery of the quantum cakes”, Kwiat and Hardy use the situation with $p_r = 3/5$ whence $p = 9/100$.

6 References

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