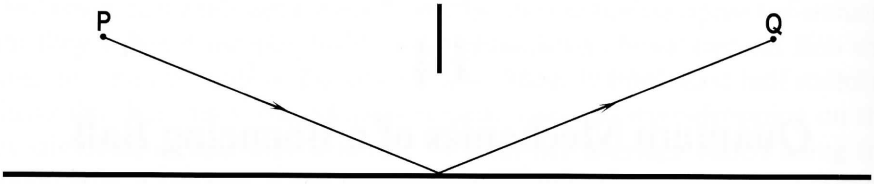
14
Quantum Mechanics of a Bouncing Ball

We started to investigate quantum mechanics by considering only the quantization of magnetic arrows. In our explorations we found out that the magnetic arrow had some funny properties (for example, it was possible that m_x did not have a definite value), but at first it seemed that other properties, such as the position of an atom, behaved in the familiar classical way. Eventually (section 9.3) we found that it was also possible to have an atom without a definite value for its position. In this chapter, we investigate what happens when we apply quantum mechanics to a particle's position.

14.1 Ball bouncing from a floor

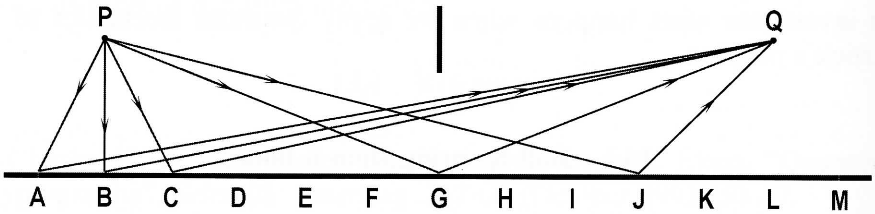
This chapter will show our framework for quantum mechanics in action, by applying it to the problem of a ball bouncing from a floor. Let us use a very fast ball, such as an electron, so that we can ignore the force of gravity. (We restrict ourselves to an electron that is moving fast on a human scale but slow compared to the speed of light, so that relativistic considerations don't come into play. Also, the magnetic arrow associated with the electron has no effect on the phenomena described in this chapter, so I won't mention it again.)

Imagine a source of balls that could send a ball flying in any direction, for example a hot tungsten filament that boils off electrons. Suppose a ball begins at point P, bounces off the floor, and ends up at point Q. (Points P and Q are equally distant from the floor. To make sure that the ball bounces off the floor rather than goes directly from P to Q, we put a barrier half-way between the two points.) In classical mechanics this sequence can happen in only one way, namely by the ball hitting the floor midway between points P and Q.



But in quantum mechanics there are many ways of going from P to Q, and each way will contribute some amplitude to the process.* In this discussion we will ignore curlicue paths from P to Q and consider only those paths that consist of a straight line from P to some point on the floor, and then a straight line from that point to Q. We will also ignore paths that go out of the plane of the page. Although it's not yet obvious (see problem 14.2), these simplifications do in fact give the correct answer, and they also give a correct feel for how to do the full problem! Here are the steps (from page 78) in finding the probability to pass from P to Q.

Step 1: Enumerate all the paths from P *to* Q. The complete enumeration is difficult, because there are an infinite number of paths, so I'll just draw some representative paths and label them according to the point where they hit the floor.



The classical path is the one that hits the floor at point G, but in quantum mechanics we must consider *all* possible paths.

Step 2: Assign an amplitude arrow to each path. Here we need to be creative. You might think that the classical path is the "most important", and thus should be assigned the longest arrow. But this is not the case at all. The correct rule assigns to every path an arrow of the same magnitude, but with the different arrows pointing in different directions. The arrow assigned to a path is found by starting with an arrow pointing directly to the right and then rotating it counterclockwise according to the length of the path from P to Q:

$$\text{number of rotations} = 1.51 \times 10^{26} \times (\text{length of path in cm}) \\ \times (\text{mass of ball in grams}) \times (\text{speed of ball in cm/s}).$$

* In both classical and quantum mechanics, not all the balls starting out at point P go to point Q. None of the balls that start out simply vanish, but many of them do not go to point Q.

There is no way that you — or anyone else for that matter — can derive this rule. It is one of the fundamental laws of nature and cannot be derived from anything simpler.

> *Technical aside:* Throughout this book I have tried to be non-technical yet completely honest and truthful. In the above formula I have had to retreat somewhat from my principled stance. It is true only for particular circumstances, and I don't know how to describe these circumstances precisely without invoking technical terms. The formula's limitation involves the fact that it purports to give the amplitude for moving from point P to point Q, whereas what's really needed is the amplitude for moving from point P at time t_p to point Q at time t_q. A symptom that the formula suffers from illness is that it invokes a speed for moving between two positions and, as we will see in section 14.3, a ball cannot have a definite position and a definite speed at the same time. While I'm off on a technical aside, let me point out that the formula above is called the "de Broglie relation", and the number 1.51×10^{26} which appears in the formula is called the inverse of Planck's constant h.

The number of rotations may, of course, be a fraction. For example, 13.5 rotations would result in an arrow pointing to the left, while 182.75 rotations would result in an arrow pointing downward.

Since the paths have a variety of different lengths their associated arrows point in a variety of different directions. Figure 14.1 shows how the length varies for different paths, and the arrow below each representative path shows the amplitude arrow assigned to that path.

Notice that path A is considerably longer than path B so the arrow associated with path A has rotated much more than has the arrow associated with path B. However, path F is only a bit longer than path G, so the associated arrows are nearly parallel.

Step 3: Add up all the arrows. This seems like a formidable task, because we have to add up an infinite number of arrows! We set about doing it using the tried-and-true scheme of "divide and conquer". We will first add up the arrows over bundles of nearby paths, and only then will we find the grand total by adding up the sums associated with each bundle.

Consider a bundle of paths like A and B and C, where the arrow changes direction dramatically from one path to another. The arrows assigned to individual paths point first up, then down, now right, now left, so that when they are added together their sum hardly amounts to anything. But now consider the bundle of paths F and G and H. Here the arrows are nearly parallel, so they add together rather than cancel out. You can see

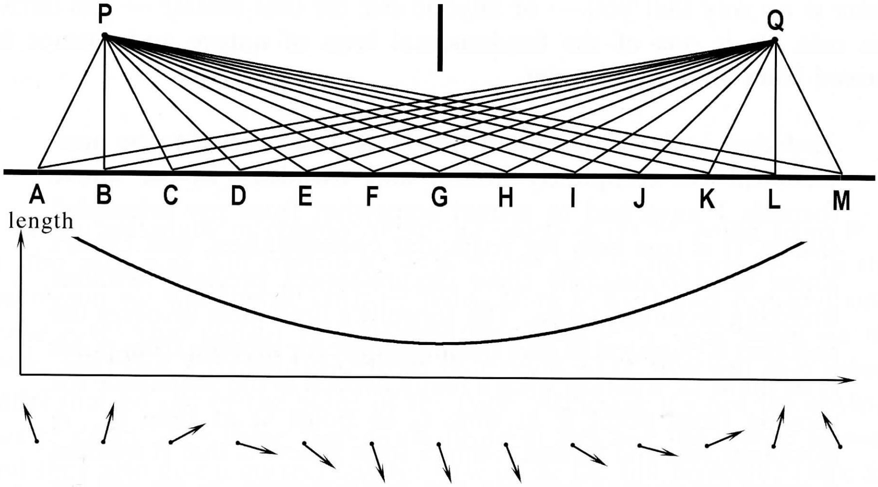


Fig. 14.1. Representative paths from P to Q, their lengths, and the amplitude arrow associated with each.

that the grand total amplitude comes almost entirely from the bundle of paths near the midpoint G, and that all the rest of the bundles contribute very little: the corresponding pieces of floor might as well be chopped up and tossed out the door. This is precisely in accord with everyday experience, from which we know that only the midpoint of the floor is needed to bounce a ball from P to Q.

14.2 Ball bouncing from a floor with holes

So you see that quantum mechanics confirms your classical expectation that the ball hits the floor only at the midpoint. It is, however, a hollow victory to work so hard to obtain a result known to every child. Can we salvage anything new or surprising from this discussion? Indeed we can.

Let us chop up and toss out the right hand three-quarters of the floor, leaving only the part near points A, B, and C. In classical mechanics it is impossible to bounce a ball from point P to point Q using this remaining piece of lumber, but in quantum mechanics we can trick the ball into bouncing this way! Remember that the total arrow associated with the bundle of paths encompassing A and B and C is nearly zero because the individual arrows are pointing every-which-way: many tilt towards the right, but just as many tilt towards the left. The trick is to remove those parts of the floor responsible for arrows that tilt towards the left. What remains will be a series of slats rather than a solid floor, but the paths bouncing off the slats all have rightward tending arrows. There will be fewer paths from P to Q, but the arrows associated with those paths, instead of cancelling out, will instead add together cooperatively to produce a substantial total amplitude arrow, and hence a substantial probability of bouncing from P to Q. (It may seem strange to get more bouncing from less floor, but I suspect that by now nothing can shock you.)

If you examine this scheme quantitatively you will find that the slats must be separated by distances of about 10^{-8} cm. It is quite difficult to mechanically produce such closely spaced slats, but fortunately nature has provided exactly the desired bouncer: it is a crystal. The rows of atoms in a crystal act as bouncing slats, while the gaps between them act as the spaces. The bouncing of electrons off a crystal (technically called "electron diffraction") was first observed by Clinton Davisson and Lester Germer in 1927.

14.3 Wave-particle duality

We have seen, in some detail now, how balls behave in quantum mechanics, and you know that this behavior is utterly unlike the behavior of classical baseballs and marbles. Just as a magnetic arrow with a definite value of m_z does not have a definite value of m_x, so an electron between release and detection does not have a definite value for its position. This means exactly what it says: it does not mean that the electron has a definite position which is changing rapidly and unpredictably, nor does it mean that the electron has a definite position but that we don't know what it is. It means that the electron just doesn't have a position, in exactly the same way that love doesn't have a color.

We found in chapter 4, "The conundrum of projections", that an atom's magnetic arrow could have a definite value for the projection m_z or a definite value for the projection m_x but not definite values for both at the same time. A similar statement turns out to be true for an electron: it can have a definite position or it can have a definite speed, but it cannot have both a definite position and a definite speed. There is no way for you to derive this — I'm just telling you. In fact, it is a technical detail that I ordinarily wouldn't mention to a general audience at all, but this detail has taken hold of the public imagination so effectively that many believe it to be the central, or perhaps even the only, principle of quantum mechanics. This detail is called the "Heisenberg uncertainty principle". (The term "uncertainty" actually reinforces the misconception that an electron has a definite position and a definite speed, but that we are not sure what they are. For this reason, the principle is more accurately called the "Heisenberg indeterminacy principle".)

The amplitude arrow picture first came up in association with waves, yet in quantum mechanics it describes the motion of a particle. This combination is sometimes called "wave-particle duality" or by saying "in quantum mechanics, an electron behaves sometimes like a wave and sometimes like a particle". I find such phrases unhelpful and extremely distasteful. From the world of everyday observation, we know about several classes of entities: marbles, putty balls, pond ripples, ocean breakers, clouds, sticks, balloons, etc. To insist that quantal entities must fall into one or another of these categories is utterly parochial. It is like a man born and raised in England who knows of several species of animals: horses, cows, pigs, etc. He travels to Africa and sees a hippopotamus, but he refuses to accept that this is a new species of animal, maintaining that it is instead an animal "in some ways like a horse and in some ways like a pig". Rather than say "an electron behaves somewhat like a wave and somewhat like a particle", I like to say "an electron behaves exactly like an electron — this behavior is not familiar and you might not be comfortable with it, but that is no reason to denigrate the electron".

> *Technical aside:* When the delayed choice experiments of section 12.2 (page 96) are adapted to atom interferometers, they reinforce the idea that an atom passes through an interferometer not as a classical particle nor as a classical wave, but rather in its own inimitable quantum mechanical fashion.

The Heisenberg uncertainty principle and the phrase "wave-particle duality" are treated with reverence and awe in some circles. But when you get right down to it they really mean nothing more than that an electron is not a small hard marble.

14.4 References

This chapter is inspired by the treatment in

R.P. Feynman, *QED: The Strange Theory of Light and Matter* (Princeton University Press, Princeton, New Jersey, 1985) pages 36–49.

A modern perspective on the Heisenberg uncertainty principle and wave-particle duality is presented by

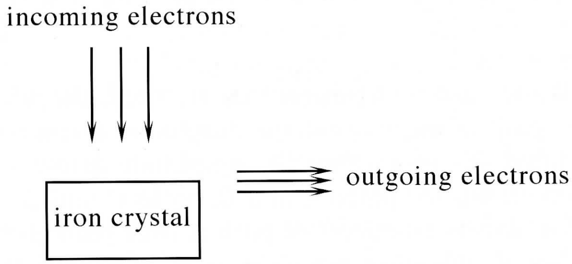
Berthold-Georg Englert, Marlan O. Scully, and Herbert Walther, "The duality in matter and light", *Scientific American*, **271** (6) (December 1994) 86–92.

The serendipitous history of the Davisson–Germer experiment is told in

R.K. Gehrenbeck, "Electron diffraction: fifty years ago", *Physics Today*, **31** (1) (January 1978) 34–41. (Be sure to notice also this issue's stunning cover photograph.)

14.5 Sample problem

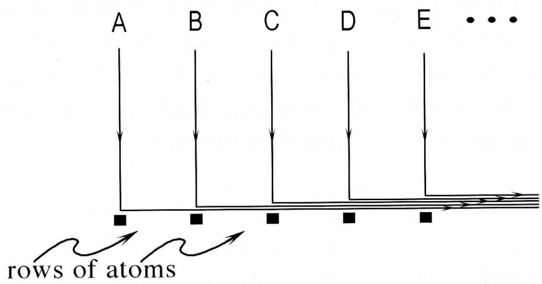
Electrons are shot down toward a crystal of iron. At what speed should they be shot so that a significant number of them are deflected by 90°? (The distance between rows of atoms in iron is 2.87×10^{-8} cm; the mass of an electron is 9.11×10^{-28} gram.)



Solution

As usual, we follow the steps listed on page 78.

Step 1: Enumerate all the paths from input to output.

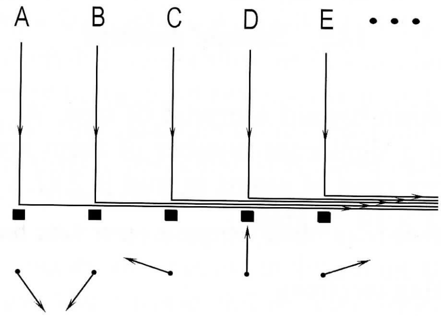


Step 2: Assign an amplitude arrow to each path. According to the formula on page 104, the arrow associated with a path rotates this many times:

$$1.51 \times 10^{26} \times (\text{length in cm}) \times (\text{mass in grams}) \times (\text{speed in cm/s}).$$

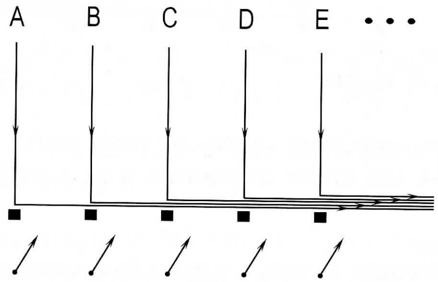
Because each path has a different length, each arrow will rotate a different amount.

Step 3: Add up all the arrows. Most circumstances are rather like the one illustrated below. (The amplitude arrow associated with each path is sketched below the letter labeling that path.)



For an electron shot down at this particular speed, the additional length of path A over path B means that the amplitude arrow associated with path A has rotated 70° more than the amplitude arrow associated with path B. The same holds for paths B and C, paths C and D, etc. (I will call this quantity the "excess rotation" of path A over path B, of path B over path C, etc.) Thus the different arrows associated with the many different paths are pointing every-which-way, so when the arrows are added they will mostly cancel out. In such circumstances, the sum arrow will be small and there will be a low probability of deflection by 90°.

Suppose, however, that an electron is shot down faster than the one above was. Then each arrow rotates more than it did above. More importantly, the excess rotation of one path over its shorter neighbor also increases. For a slight increase in speed, there will be a slight increase in excess rotation: say from 70° to 90°. Still, the arrows will be pointing every-which-way and, when added, they will mostly cancel out. But what if there is a significant increase in speed leading to a significant increase in excess rotation, say to 360°, a full rotation?



In this case each of the arrows points in *exactly* the same direction, so when they are all added together they produce a large sum arrow and hence a high probability of deflection by 90°.

What is this special speed that results in a large probability of deflection by 90°? It is the speed at which the distance between rows of atoms corresponds to exactly one rotation, that is, the speed at which

$$1 = 1.51 \times 10^{26} \times (\text{distance between rows in cm}) \times (\text{mass in grams}) \times (\text{speed in cm/s.}).$$

Solving this equation for the speed gives 2.53×10^8 cm/s. This speed is very large on a human scale, but because electrons have so little mass it is easy to make them go this fast.

14.6 Problems

14.1 *Other speeds.* The previous sample problem (section 14.5) finds a speed that gives rise to a substantial probability of deflection by 90° when an electron is shot down at an iron crystal. Will there be a substantial probability of deflection by 90° if an electron of twice this speed is used? Three times? Half the speed? One-third? One-quarter?

14.2 *Curlicue paths.* Consider the motion of a ball from point P to point Q without floors or barriers. Enumerate typical paths between the two points, including curved and three-dimensional paths, and draw representative amplitude arrows associate with each one. Generalize the reasoning on page 105 to show that, in the classical limit, the bundle of paths that are nearly straight lines from P to Q provide most of the amplitude to go from P to Q.

14.3 *Heisenberg uncertainty principle.* In his 1993 Oersted Medal acceptance speech, the distinguished physicist Hans Bethe said

> The [Heisenberg] uncertainty principle simply tells us that the concepts of classical physics are not applicable to the atomic world. But we think in classical terms, and therefore we need the uncertainty principle to reconcile our classical terms with the reality of quantum theory.

Would this passage be improved by replacing the phrase "classical terms" with "classical terminology"? Justify your answer.

14.4 *Wave-particle duality.* On page 57 I summarized the first half of this book by saying that

> If m_z has a definite value, then m_x doesn't have a value. If you measure m_x, then of course you find some value, but no one (not even the atom itself!) can say with certainty what that value will be — only the probabilities of measuring the various values can be calculated.

Produce a corresponding statement that applies to an electron rather than to the magnetic arrow of a silver atom, and that uses "position" and "speed" rather than "m_z" and "m_x".