5
Probability

I interpreted the repeated measurement experiments of the previous chapter by saying that quantum mechanics can find probabilities only, not certainties (that is, that quantum mechanics is "probabilistic", not "deterministic"). You may object, maintaining that the world is deterministic, but that my particular deterministic scheme (the "magnetic arrow") is incorrect. The next chapter presents an ingenious argument, invented by Einstein, which shows that *no* local deterministic scheme could give the results observed by experiment. In order to understand that argument you need some background in probability.

But in fact, a knowledge of probability is generally useful in day-to-day life as well as in physics. You walk across a street — what is the probability of your being hit by a car? You are advised to undergo elective surgery — what is the probability that the surgery will extend your life, and what is the probability that the surgery will go wrong and injure you? You breath some asbestos or smoke a cigarette — what is the probability of contracting cancer? Misconceptions about probability abound and can lead to disastrous public policy decisions.* A knowledge of quantum mechanics is good for your soul, but it is of practical importance only to the designers of lasers, transistors, and superconductors. A knowledge of probability is of practical importance to everyone.

* Suppose that, in a democratic society, 70% of the citizens prefer to drink beer and 30% prefer to view artwork. Does this imply ("majority rules") that all art museums should be converted into bars? Does it imply that the ratio of bars to art museums ought to be fixed by law at 7 to 3? Of course it implies neither. But many policy makers seem never to have learned this simple lesson in probability.

5.1 Gambling probability

If you toss a die, the probability of rolling a **2** is $\frac{1}{6}$. If you flip a coin, the probability of getting **heads** is $\frac{1}{2}$. In general, for gambling probabilities,

$$\text{probability of a success} = \frac{\text{number of successful outcomes}}{\text{number of possible outcomes}}.$$

This rule holds only for gambling probabilities like those we have just mentioned. It does not apply, for example, to surgery, where there are only two possible outcomes — survival and death — but the chance of survival is far greater than $\frac{1}{2}$. Nor does it apply to the Stern–Gerlach analyzer, where an incoming atom can leave through either the + exit or through the − exit, but figure 4.1 shows that the probabilities of these two possible outcomes are not always 50%. Finally, if you buy a lottery ticket, there are only two possible outcomes — winning and losing — but the probability of winning is sadly less than $\frac{1}{2}$.

5.2 Compound probabilities

Example 1. Toss a die. What is the probability of rolling either a **1** or a **3**? In this case, there are six possible outcomes and two successful outcomes, so the probability of success is $\frac{2}{6} = \frac{1}{6} + \frac{1}{6}$. In general, the word "or" is a signal to *add* probabilities.

Example 2. Toss a die and simultaneously flip a coin. What is the probability of getting **2** and **tails**? In this case there are twelve possible outcomes (**1** and **heads, 1** and **tails, 2** and **heads, 2** and **tails**, and so forth up to **6** and **tails**) so the probability of getting **2** and **tails** is $\frac{1}{12} = \frac{1}{6} \times \frac{1}{2}$. In general, the word "and" is a signal to *multiply* probabilities.

Example 3. Flip three coins (or flip one coin three times):

possible outcome	probability of this outcome	number of heads
HHH	1/8	3
HHT	1/8	2
HTH	1/8	2
THH	1/8	2
HTT	1/8	1
THT	1/8	1
TTH	1/8	1
TTT	1/8	0

Thus the probability of obtaining three heads is $\frac{1}{8}$, of two heads is $\frac{3}{8}$, of one head is $\frac{3}{8}$, of no heads is $\frac{1}{8}$.

Example 4. Toss two dice (or toss one die two times). What is the probability that the sum of the face-up dots is four? If the first die lands on **4** and the second on **3**, I will call the outcome "[**4**, **3**]".

probability of [**1**, **1**] is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
probability of [**1**, **2**] is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
probability of [**2**, **1**] is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
probability of [**1**, **3**] is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
probability of [**3**, **1**] is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
probability of [**2**, **2**] is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
and so forth to
probability of [**5**, **6**] is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
probability of [**6**, **5**] is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
probability of [**6**, **6**] is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Now

probability that the sum tossed is four =
(probability of [**2**, **2**]) +
(probability of [**1**, **3**]) +
(probability of [**3**, **1**]),

but

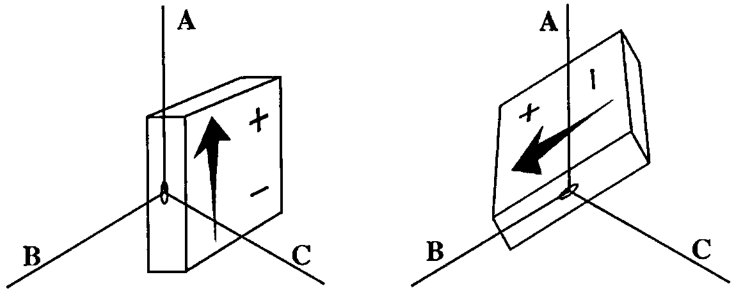
(probability of [**2**, **2**]) = (prob. of **2**) × (prob. of **2**) = $\frac{1}{6} \times \frac{1}{6}$,
(probability of [**1**, **3**]) = (prob. of **1**) × (prob. of **3**) = $\frac{1}{6} \times \frac{1}{6}$,
(probability of [**3**, **1**]) = (prob. of **3**) × (prob. of **1**) = $\frac{1}{6} \times \frac{1}{6}$,

thus

probability that the sum tossed is four =
$(\frac{1}{6} \times \frac{1}{6}) + (\frac{1}{6} \times \frac{1}{6}) + (\frac{1}{6} \times \frac{1}{6}) = \frac{1}{12}$.

5.3 Tilting Stern–Gerlach analyzer

Mount a single Stern–Gerlach analyzer on a pivot so that it can be tilted to have its magnetic field point in any of the three directions **A**, **B**, or **C**. In the figure on the next page, it is tilted to orientation **A** on the left and to orientation **B** on the right. This analyzer is switched at random between these three orientations, each orientation having probability $\frac{1}{3}$. Suppose an atom with $m_z = +m_B$ were fed into the analyzer. (For example, the atom might have just emerged from the + exit of an analyzer fixed in orientation **A**.) What is the probability that it leaves through the + exit?



If the orientation is **A**,
the probability that the atom leaves the + exit is 1.
If the orientation is **B**,
the probability that the atom leaves the + exit is $\frac{1}{4}$.
If the orientation is **C**,
the probability that the atom leaves the + exit is $\frac{1}{4}$.

(These last two facts come from figure 4.1 on page 27: when $\theta = 120°$ or $\theta = 240°$, the probability is $\frac{1}{4}$.)
Now

probability that atom leaves from + exit =
(probability that it does so when orientation is **A**) +
(probability that it does so when orientation is **B**) +
(probability that it does so when orientation is **C**)

but

probability that it does so when orientation is **A** $= \frac{1}{3} \times 1$
probability that it does so when orientation is **B** $= \frac{1}{3} \times \frac{1}{4}$
probability that it does so when orientation is **C** $= \frac{1}{3} \times \frac{1}{4}$

thus

The probability that an atom entering with $m_z = +m_B$
leaves from the + exit is
$(\frac{1}{3} \times 1) + (\frac{1}{3} \times \frac{1}{4}) + (\frac{1}{3} \times \frac{1}{4}) = \frac{1}{2}$.

5.4 References

Warren Weaver, *Lady Luck: The Theory of Probability* (Doubleday, Garden City, New York, 1963).

Vinay Ambegaokar, *Reasoning About Luck: Probability and Its Uses in Physics* (Cambridge University Press, New York, 1996).

Harold Warren Lewis, *Why Flip a Coin? The Art and Science of Good Decisions* (John Wiley, New York, 1997).

5.5 Problems

It is important that you look at the first six problems in this section. The remaining problems are fun and informative but are not needed to support this book's train of argument.

5.1 *Three dice.* If you throw three dice, what is the probability that a total of four dots are face up?

5.2 *License plates.* Suppose that in the state of Iowa auto license plates are identified by three letters followed by three numbers, and that the numbers are chosen at random. If you glance at an Iowa plate, what is the probability that the two last digits will be the same?

5.3 *Two dice.* Throw two dice. What is the probability that the sum of the face-up dots is more than four?

5.4 *The coin toss.* Toss a single coin ten times.

(a) What is the probability of obtaining all heads (the pattern HHHHHHHHHH)?

(b) What is the probability of obtaining alternating heads then tails (the pattern HTHTHTHTHT)?

(c) What is the probability of obtaining the pattern HTTTHHTTHT?

(d) What is the probability of obtaining a pattern with one tail and nine heads?

5.5 *Tilting Stern–Gerlach analyzer.* In section 5.3 we found the probability for an atom that entered a tilting Stern–Gerlach analyzer with $m_z = +m_B$ to leave through the + exit. What is that probability for an atom that enters:

(a) With $m_z = -m_B$?

(b) With $m_{120°} = +m_B$?

(c) With $m_{(-120°)} = +m_B$?

5.6 *Military draft lottery.* From 1967 to 1972 the United States used a military draft lottery in which birthdays were selected at random, and the army drafted first men born on the first date selected, then men born on the second date selected, and so forth. Suppose a small country uses a similar system, but the country's records include not birth date, but only birth season: summer or winter. In the year 1968 there are 1000 draftable men of whom 600 were born in winter and 400 were born in summer. The military requires 700 draftees. (Thus a "fair" system would assign each man a probability 7/10 of being drafted.) The country holds a lottery to determine whether summer-born or winter-born men will be called up first (either possibility has probability 1/2). Within each birth category, the military drafts men at random. The drafting continues from the first category into the second until the requirement of 700 draftees is filled.

(a) Show that if the summer-born are called up first, the probability of a winter-born man being drafted is 1/2.

(b) Show that if the winter-born are called up first, the probability of a summer-born man being drafted is 1/4.

(c) What is the overall probability of drafting a summer-born man? A winter-born man? (The term "overall probability" means the probability that would be calculated *before* the lottery was held.)

Comment upon the fairness of this draft lottery scheme.

5.7 *Speeding tickets.* Benjamin Marrison of the Cleveland *Plain Dealer* wondered whether Ohio state troopers were more likely to hand out speeding tickets at the end of the month ("Spotting the speeders: How, when, and where troopers will get you", 3 December 1995). He uncovered data showing that for the year 1994, a total of 19 737 speeding tickets were issued on the 28th of some month, 19 623 were issued on the 30th, but only 18 845 were issued on the 31st. From this he concluded that one is actually less likely to get a speeding ticket at the end of a month. Comment.

5.8 *Outdoor hazards.* From the New York *Times* (30 June 1998): "Where are you more likely to be injured: climbing down a rock face or sitting around a campsite? ... A new study of the hazards of national parks found that injuries at campsites ... outnumber those sustained during rock climbing by more than 3 to 1." Comment.

5.9 *Winning the lottery twice.* Suppose that the state of New Jersey runs one lottery game each week, that each week there is one and only one winner, that five million individuals play each week, that each of those individuals buys a single ticket, and that this same group of individuals plays every week. (These suppositions are, of course, not precisely correct. On the other hand they are not terribly different from the actual situation, and this simplification allows the situation to be understood much more readily than a "perfect portrait" would permit.) What is the probability that:

(a) One particular player, Sylvia Struthers of Clinton Mills, New Jersey, wins in the two successive lotteries held on the ninth and tenth weeks of the year 1998.

(b) A player, not necessarily Ms. Struthers, wins in both of these two lotteries.

(c) A player wins in successive weeks in 1998.

(d) A player wins twice in 1998.

5.10 *Average vs. typical, part 1.* A politician claims that she "always stands up for the average man". Does that mean that she always supports the majority of people? Hint: How many people do you know who are of average height?

5.11 *Average vs. typical, part 2.* At the Lincoln Street branch of the First National Bank, a customer needing a teller enters the bank, on average, once each minute, and each teller transaction lasts, on average, three minutes. Based on these facts, the branch manager decides to staff the bank with exactly three tellers. Why was the branch manager fired?

5.12 *Correlations.* In a poll, one thousand individuals are asked about their preferences in music and in dining. One-tenth of the individuals preferred opera to rock, and one-fifth of them preferred French restaurants to fast food restaurants. Is the probability that one of the polled individual prefers *both* opera *and* French restaurants equal to 2% ? ($2\% = \frac{1}{50} = \frac{1}{10} \times \frac{1}{5}$.)

5.13 *Random vs. haphazard.* Smith and Jones are running for congress, and Ms. Struthers wants to know who will probably win. So she polls ten of her friends, and finds that eight plan to vote for Smith and two for Jones. She is surprised and shocked when Jones wins by a margin of 54% to 46%. What was wrong with her poll?