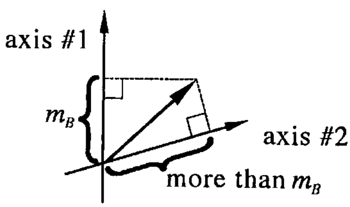
4

The Conundrum of Projections; Repeated Measurements

4.1 The conundrum of projections

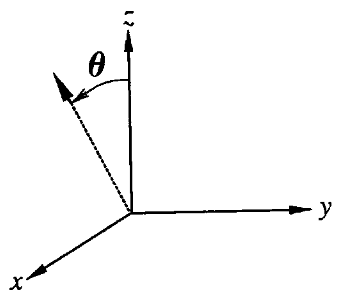
Whenever Stern and Gerlach measured the projection of a silver atom's magnetic arrow on an axis, they found either $+m_B$ or $-m_B$. But the figure below demonstrates that it is impossible for any arrow to have a projection of $\pm m_B$ on *all* axes! Even if the projection onto the vertical axis (in the figure, axis #1) happens to be $+m_B$, then we can always draw some other axis (such as axis #2) that has a different projection (in the figure, something more than m_B).



I call this difficulty the "conundrum of projections". The fact that the projections can only take on the values of $\pm m_B$ is strange and unexpected, but it's something that we can live with. (After all, much of human behavior — and most of politics — is strange and unexpected too, once you think about it.) The conundrum of projections is far more serious, because it seems at first to be not just strange, but logically impossible. In order to resolve the conundrum, we will introduce experiments in which we actually measure the projection on various axes, and we will let the results of those experiments suggest a resolution. Before doing this, however, we must introduce some terminology.

Terms for projections

The figure below shows four different axes. In this book, except for section 11.1, we will consider only projections onto axes lying within the (x, z) plane.



The projection of the arrow onto a vertical axis is called m_z.
The projection of the arrow onto a horizontal axis is called m_x.
The projection of the arrow onto a downward vertical axis is called $m_{(-z)}$.
The projection of the arrow onto an axis within the (x, z) plane but tilted at an angle θ to the vertical is called m_θ. (Thus $m_z = m_{0^\circ}$, $m_x = m_{90^\circ}$, and $m_{(-z)} = m_{180^\circ}$.)

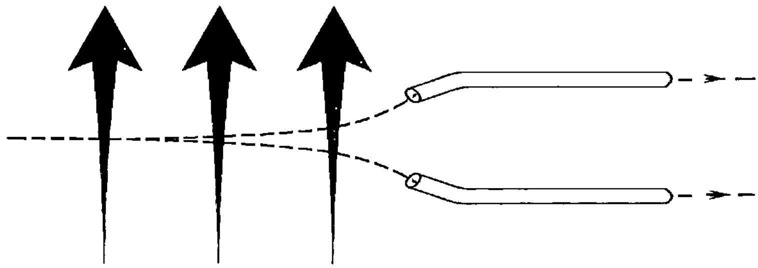
Note that if the projection onto some axis is $+m_B$, then the projection onto an axis pointing in the opposite direction is $-m_B$.

Stern–Gerlach analyzer

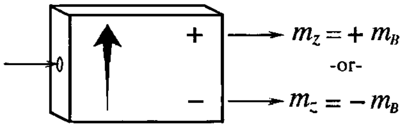
For convenience, I will package the Stern–Gerlach apparatus into a tall thin box and call it a *Stern–Gerlach analyzer*. There are only two places where a silver atom can come out of the apparatus, so the analyzer box has only two exit ports. (In the rest of this book, I will use only silver atoms and I will usually call them just "atoms" rather than "silver atoms".) The box also contains plumbing to the right of the non-uniform magnetic field which pushes the atoms around so that an outgoing atom follows a track parallel to the track of an incoming atom.* This plumbing doesn't affect the atom's magnetic arrow. These alterations do not change any important property of the Stern–Gerlach apparatus; they merely make our diagrams easier to read.

* One way to produce such plumbing is by installing a second non-uniform magnetic field that points in the opposite direction from the first.

In summary, the raw apparatus shown here:



is packaged into a box and represented as:

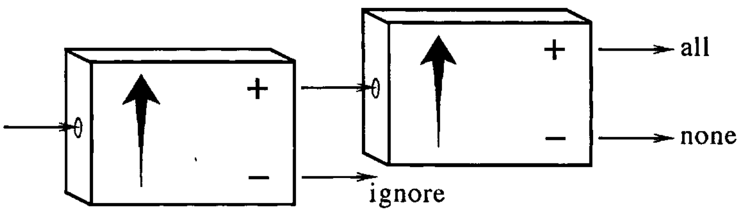


An atom enters the box on the left, and then either it leaves through the upper exit, marked +, in which case it has $m_z = +m_B$, or else it leaves through the lower exit, marked −, in which case it has $m_z = -m_B$.

On the other hand, if the Stern–Gerlach analyzer were oriented horizontally, then the exiting atom would have either $m_x = +m_B$ or else $m_x = -m_B$. Or, we could tilt the Stern–Gerlach analyzer box 17° to the right of vertical, in which case exiting atoms would have either $m_{17°} = +m_B$ or else $m_{17°} = -m_B$. In other words, a vertical analyzer measures m_z, a horizontal analyzer measures m_x, and our tilted analyzer measures $m_{17°}$.

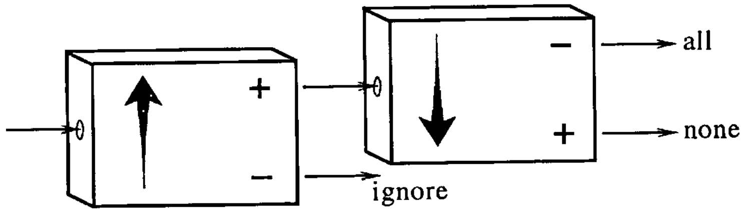
4.2 Repeated measurement experiments

Experiment 4.1. Measurement of m_z, then m_z again.



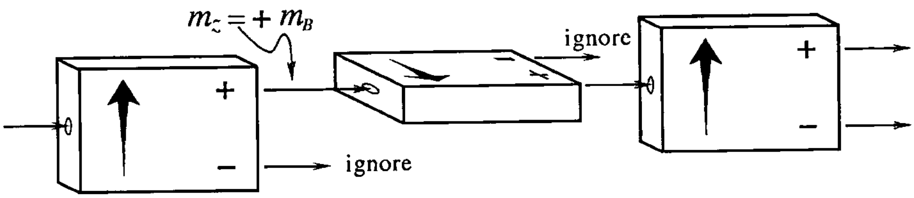
An atom entering the first analyzer will leave either the top (+) exit or the bottom (−) exit. In the latter case the exiting atom has $m_z = -m_B$ and we ignore it. In the former case the exiting atom has $m_z = +m_B$ and it is fed into the second analyzer. All such atoms leave the + exit of the second analyzer. In short, if an atom is found to have $m_z = +m_B$ at the first analyzer, then it does at the second analyzer as well.

Experiment 4.2. Measurement of m_z, then $m_{(-z)}$.



If an atom is found to have $m_z = +m_B$ at the first analyzer, then it has $m_{(-z)} = -m_B$ at the second analyzer.

Experiment 4.3. (The crucial experiment.) Measurement of m_z, then m_x, then m_z.



An atom entering analyzer A will leave from either the + exit or the − exit. In the latter case we ignore it, and in the former case we feed the atom (with $m_z = +m_B$) into analyzer B, a horizontal Stern–Gerlach analyzer that measures m_x. The atom will then either leave the − exit (in which case it has $m_x = -m_B$) and we ignore it, or else it will leave the + exit (in which case it has $m_x = +m_B$) and we feed it into analyzer C, a vertical analyzer that measures m_z.

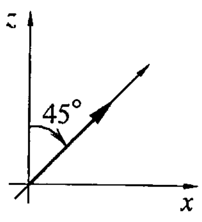
What do you think will happen then? You might reason that this atom is known to have $m_z = +m_B$, because it left the + exit of analyzer A (as well as $m_x = +m_B$, because it left the + exit of analyzer B) and thus that it will leave the + exit of analyzer C, just as the atoms in experiment 4.1 did. This seems reasonable, and I will call it the "good guess argument". But in fact this *does not* happen. Instead, some atoms at this stage leave the + exit of analyzer C and others leave the − exit.

In summary, when an atom enters analyzer B it has a definite value of m_z, namely $+m_B$ — we know this because of experiment 4.1. But when that atom leaves analyzer B it *does not* have a definite value of m_z — we know this because when it enters analyzer C it might leave through either the + or the − exit.

It's worth investigating this unexpected result further. We perform the experiment many times, and each time record whether the atom entering analyzer C leaves through the + exit or through the – exit. We find that there is no regular pattern to the exits, but that about half of the atoms leave through + and the rest leave through –. Thus although we cannot say with certainty which way the atom will leave analyzer C, we can say that it has probability one-half of leaving through either exit.

The following picture helps some people. They think of an atom leaving the + exit of analyzer B as having a magnetic arrow that points straight out of the page (that is, in the $+x$ direction). In classical mechanics, if such an atom entered analyzer C it would pass straight through. But the Stern–Gerlach result shows that in truth (that is, in quantum mechanics) it can't pass straight through — it must go either up or down (that is, it must leave through either the + exit or the – exit). If you "want" to go straight but are forced to go either up or down, the best you can do is go up half the time and down half the time. This picture is not entirely accurate (as we will see in detail later) but if you keep in mind both the picture and its limitations it may help you visualize the process.

I want to go back for a moment to the good guess argument, the one which suggests that every atom should leave analyzer C through the + exit. Experiment shows that this result is not correct, but we can also produce reasoning showing that it is not correct: We know that an atom leaving the + exit of analyzer B has $m_x = +m_B$. The good guess argument supposes that, by virtue of having previously left the + exit of analyzer A, it also has $m_z = +m_B$. You can see from the diagram below that an atom with both $m_x = +m_B$ and $m_z = +m_B$ would have a value for m_{45° that is *bigger* than m_B. (Experts in geometry will recognize from the diagram



that in fact $m_{45^\circ} = +\sqrt{2}m_B$, but you don't need to be an expert to see that m_{45° is larger than m_B.) But whenever m_{45° is measured, it is found to be either $+m_B$ or $-m_B$, and never to be bigger than $+m_B$!

An atom with a definite value for both m_x and m_z would have values for other projections that are not $\pm m_B$, and such atomic states do not exist. The flaw in the good guess argument is not in its reasoning, but in its assumptions. It assumed that an atom leaving analyzer B would have the same value of m_z as it did when it entered, and this is false.

4.3 The upshot

We escape from the conundrum of projections via probability. If an atom has a definite value of the projection of its magnetic arrow on one axis, then it does *not* have a definite value of the projection of its arrow on some other axis. Given an atom with $m_x = +m_B$, to ask "What is the value of m_z?" is just like asking "What is the color of love?". These questions have no answers because for this atom, m_z *doesn't have* a value in just the same way that love doesn't have a color. What *can* be said of such an atom is the probability of finding either of the two possible projections on the vertical axis.

Terminology note: Be wary of the phrase "definite value". When I say "An atom with a definite value of m_x doesn't have a definite value of m_z" what I really mean is "An atom with a value of m_x doesn't have a value of m_z". The second wording is more accurate and more clearly points out the difference between the quantal world and the classical world. But it is so stark that it makes most physicists uncomfortable. (It certainly makes *me* uncomfortable.) So usually I will employ the euphemism of "definite value". This is a personal failing of mine but I can't help it.

If the second analyzer were tilted at an angle θ relative to the first, then what would be the probability that an atom leaving the + exit of the first analyzer will leave the + exit of the second? We have so far discussed the situations $\theta = 0°$, 90°, and 180° (in experiments 4.1, 4.3, and 4.2 respectively). In these situations the answers were 1, $\frac{1}{2}$, and 0. The experimentally determined answer to the question for any value of θ is given in figure 4.1. Notice that the graph interpolates smoothly between the known results at $\theta = 0°$, 90°, and 180°. For example, at $\theta = 60°$ the probability is $\frac{3}{4}$. (Experts in trigonometry will have already guessed the truth, namely that the probability is given by $\cos^2(\theta/2)$.)

4.4 Barriers to understanding

We have already reached the first central concept of quantum mechanics: *The outcome of an experiment cannot, in general, be predicted exactly; only the probabilities of the various outcomes can be found.* Many learners find their grasp beginning to slip at this point. If you are one of them, then don't flounder, but instead look inside of yourself to find the reason.

Is it that you hate math, so when I wrote down $\cos^2(\theta/2)$ you felt nauseous? Then relax: you'll never need to calculate with cosines.

Is it because you don't care about magnetic needles and don't want to learn about them? Then remember that I'm using magnetic needles only as an example to illustrate the principles of quantum mechanics, and that those principles describe all the actions in the universe.

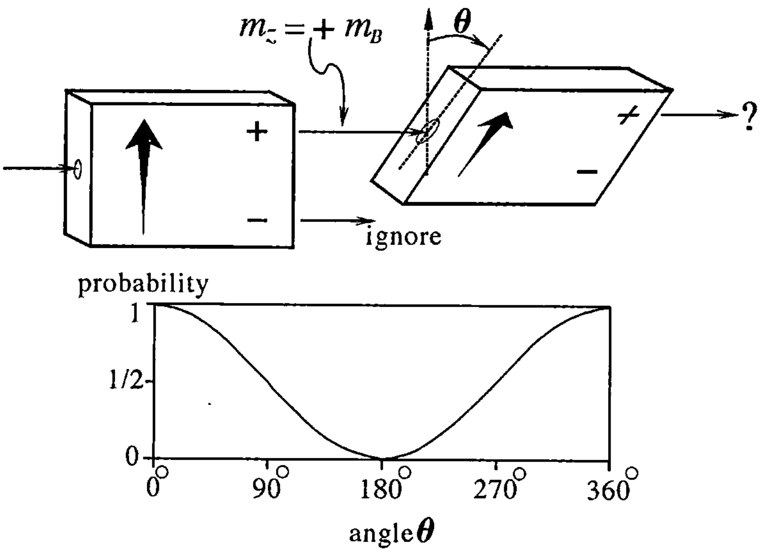


Fig. 4.1. The probability of an atom leaving the + exit of the second analyzer as the tilt angle θ between the two analyzers is varied. The probability is 1 for $\theta = 0°$, $\frac{3}{4}$ for $\theta = 60°$, $\frac{1}{2}$ for $\theta = 90°$, $\frac{1}{4}$ for $\theta = 120°$, 0 for $\theta = 180°$, etc.

A deeper problem bothers those who say "I see *that* only probabilities can be found, but I want to know *why* only probabilities can be found." Fundamentally, I have no answer to this concern. I don't know why the universe works the way it does any more than you do. I'm not God, I didn't create the universe, so don't complain to me. However, I suspect that when you ask the question "why?", you're really worried about something else. There are lots of good "why" questions that you never ask: Why does the universe have three dimensions? Why do we eat pancakes often for breakfast but rarely for dinner? Why do women wear skirts and men pants? You don't ask these questions because you're so familiar with the facts that you never stop to question why they're true. I think that most people who ask "Why can only probabilities be found?" are really just crying out that the new world of quantum mechanics is strange and unfamiliar. It certainly is. But this should be seen as a challenge to invite exploration rather than an excuse to crawl back into your familiar, secure, classical hole.

Finally, the most dangerous barrier to understanding of all: You don't want the result to be true. It seems strange — it *is* strange — so you simply reject it. But all sorts of things seem strange upon first encounter. When it was first discovered that the earth was round, that must have seemed strange too! I admit that even though I have studied a lot of quantum mechanics, it still seems strange to me, but it seems strange and delightfully quirky, rather than strange and repulsive. If you are rejecting quantum mechanics simply because it's strange, then I urge you to keep at it until you find it as beautiful as I do.

4.5 Sample problem

In experiment 4.3, half the atoms entering analyzer C leave through the + exit and half leave through the − exit. Suppose the experiment is altered by tilting analyzer A 30° to the right of vertical. Analyzers B and C are not changed. In this new experiment, what portion of the atoms entering analyzer C will leave through the + exit?

Solution

An atom leaving the + exit of analyzer B has $m_x = +m_B$. It doesn't care what state it was in when it entered analyzer B — it could have come directly from an oven, or it could have come through a complicated set of a dozen analyzers tilted at various angles — the output state is specified completely by saying $m_x = +m_B$. Thus half of the atoms entering analyzer C will leave through the + exit whether analyzer A is tilted to 30°, 0°, or any other angle.

4.6 Problems

4.1 *The conundrum of projections.* An arrow is three inches long and points due west. What is its projection on an axis that points: (a) due west, (b) due east, (c) due north, (d) straight up, (e) straight down, (f) half-way between straight up and due west?

4.2 *Two analyzers.* In experiment 4.1 on page 23 the atoms leaving the − exit of the first analyzer were ignored. What would happen to them if they were instead fed into another vertical analyzer?

4.3 *Certainty.* I have claimed that "the outcome of an experiment cannot, in general, be predicted exactly; only the probabilities of the various outcomes can be found". Yet in experiment 4.1 on page 23 an atom entering the second analyzer will certainly leave through the + exit. How can my claim and this experiment be reconciled?

4.4 *The state of an atom.* Which phrase best describes the state of an atom that leaves the + exit of analyzer B in experiment 4.3 on page 24: (1) It has $m_z = +m_B$. (2) It has $m_x = +m_B$. (3) It has both $m_z = +m_B$ and $m_x = +m_B$.

4.5 *Three analyzers.* In experiment 4.3 on page 24 the atoms leaving the − exit of analyzer B were ignored. What would happen to them if they were instead fed into another horizontal analyzer? Into a vertical analyzer?

4.6 *Three analyzers rearranged.* In experiment 4.3 on page 24, atoms leave the three analyzers according to statistics described in this table:

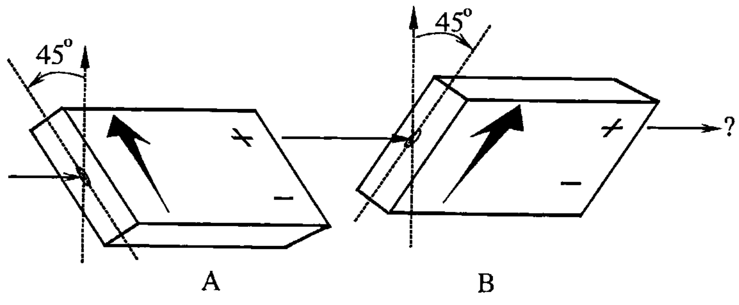
analyzer	exit statistics
A	depends on character of incoming atoms
B	half through +, half through −
C	half through +, half through −

If analyzer A were lifted so that the atoms entering analyzer B came from the − exit of A (rather than from the + exit), how would the table change?

4.7 *Rotations.* Would any of the results in this chapter change if the entire experimental apparatus (source, all analyzers, and detectors) were rotated as a unit?

4.8 *Different angles.* A careful reading of the graph in figure 4.1 shows that if the first analyzer is vertical, and the second is tilted to the right of vertical by 60°, then the probability of an atom leaving the + exit of the second analyzer is $\frac{3}{4}$. What would be the probability if the first analyzer were 60° to the *left* of vertical and the second were vertical?

4.9 *More different angles.* Two Stern–Gerlach analyzers are arranged as shown below. Analyzer A is tilted 45° to the left of vertical, while analyzer B is tilted 45° to the right of vertical. Atoms leaving the + exit of A are fed into the input of B. What is the probability that an atom entering B will leave it through the + exit?



Hint: What would happen if A were tilted 10° to the left, and B were tilted 80° to the right?

4.10 *Three analyzers with different angles.* Consider experiment 4.3 on page 24, but suppose that analyzer B were not horizontal, but rather tilted to the right of vertical by 60°. In this case, what is the probability that an atom entering analyzer C will emerge from the + exit? From the − exit? Hint: See figure 4.1.

4.11 *Barriers to understanding.* Distinguish between "a description of the rules of chess", "an understanding of the rules of chess", and "an explanation for the rules of chess". Which of these do you need to play a good game of chess?

4.12 *Familiar vs. understood.* My mother once told me that "I used to understand telephones, but I don't understand these new cellular phones." When I asked her how a conventional telephone worked, she could only say "I think it has carbon in it." In three or fewer sentences, show how this story illustrates the difference between familiarity and understanding. (My mother is, by the way, perfectly capable of using any sort of telephone.)

4.13 *Explaining Newtonian and quantum mechanics.* (For technical readers.)

(a) In Newtonian mechanics, force is related to acceleration ($\mathbf{F} = m\mathbf{a}$), whereas most laymen believe that force is related to speed. (Lay belief: "If you push something, it moves." Newton: "If you push something, its motion changes.") How would you respond to an intelligent layman who asked you why Newtonian mechanics is correct?

(b) In classical mechanics, the future can be predicted exactly, whereas in quantum mechanics only probabilities can be found. How would you respond to an intelligent layman who asked you why quantum mechanics is correct?