White dwarf stars

a. $N = \dfrac{M}{m_e + m_p} \approx \dfrac{M}{m_p}$.

b. The exact gravitational self energy is

$$\tfrac{1}{2}\int d^3r_1 \int d^3r_2 \left(-\frac{G\rho(\vec{r}_1)\rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}\right).$$

By dimensional analysis, an estimate for the gravitational self energy of a sphere of radius R, mass M, and unknown mass distribution is

$$-c\frac{GM^2}{R}.$$

c.

$$\text{KE of ground state } = \tfrac{3}{5}N\mathcal{E}_F = \tfrac{3}{5}N\frac{\hbar^2 k_F^2}{2m_e}$$

$$k_F = \left(3\pi^2\frac{N}{V}\right)^{1/3} = \left(3\pi^2\frac{N}{(4/3)\pi R^3}\right)^{1/3} = \frac{1}{R}\left(\frac{9\pi}{4}N\right)^{1/3}$$

$$\begin{aligned}\text{KE} &= \frac{3}{10}\left(\frac{9\pi}{4}\right)^{2/3}\frac{\hbar^2}{m_e}\frac{N^{5/3}}{R^2} \qquad [\![\ldots \text{use } N = M/m_p \ldots]\!]\\ &= \frac{9}{20}\left(\frac{3\pi^2}{2}\right)^{1/3}\frac{\hbar^2}{m_e m_p^{5/3}}\frac{M^{5/3}}{R^2}\end{aligned}$$

d.

$$c\frac{GM^2}{2R} = \frac{9}{20}\left(\frac{3\pi^2}{2}\right)^{1/3}\frac{\hbar^2}{m_e m_p^{5/3}}\frac{M^{5/3}}{R^2}$$

$$\begin{aligned}RM^{1/3} &= \frac{2}{cG}\left(\frac{9}{20}\right)\left(\frac{3\pi^2}{2}\right)^{1/3}\frac{\hbar^2}{m_e m_p^{5/3}} \qquad [\![\ldots \text{use } c = 3/5\ldots]\!]\\ &= \frac{3}{2}\left(\frac{3\pi^2}{2}\right)^{1/3}\frac{\hbar^2}{m_e m_p^{5/3}}\\ &= 2.87 \times 10^{17}\ \text{m kg}^{1/3}\end{aligned}$$

e. 22 700 km (compare the sun's radius of 696 000 km, the earth's radius of 6 380 km)

f. For neutron stars, the derivation is identical except that m_e and m_p are both replaced by the mass of a neutron, which is nearly equal to m_p. Thus for a neutron star,

$$RM^{1/3} = (2.87 \times 10^{17}\text{m kg}^{1/3})\left(\frac{m_p}{m_e}\right).$$

Applying this relationship to a neutron star with the mass of the sun gives a radius of 12.4 km.