Volume of a d-dimensional sphere

Important fact from reading:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad \text{when } p,q > 0.$$

a.

$$\begin{aligned}
\frac{C_{d+1}}{C_d} &= \int_{-1}^{1} (1-u^2)^{d/2}\,du && \text{(the integrand is even so...)} \\
&= 2\int_0^1 (1-u^2)^{d/2}\,du && \text{(use the substitution } x = u^2\text{)} \\
&= 2\int_0^1 (1-x)^{d/2} \frac{1}{2\sqrt{x}}\,dx \\
&= \int_0^1 x^{-1/2}(1-x)^{d/2}\,dx \\
&= B(\tfrac{1}{2}, \tfrac{d}{2}+1)
\end{aligned}$$

b.

$$C_{d+1} = C_d \frac{\Gamma(\frac{1}{2})\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{2}+\frac{3}{2})} = C_d\left[\sqrt{\pi}\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{2}+\frac{3}{2})}\right]$$

Work out the first few C_d coefficients, starting from the well-known result $C_2 = \pi$:

$$\begin{aligned}
C_2 &= \pi \\
C_3 &= \pi\left[\sqrt{\pi}\frac{\Gamma(\frac{4}{2})}{\Gamma(\frac{5}{2})}\right] = \frac{\pi^{3/2}}{\Gamma(\frac{5}{2})} && \text{(Using } \Gamma(2) = 1! = 1.\text{)} \\
C_4 &= \frac{\pi^{3/2}}{\Gamma(\frac{5}{2})}\left[\sqrt{\pi}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{6}{2})}\right] = \frac{\pi^{4/2}}{\Gamma(\frac{6}{2})} \\
C_5 &= \frac{\pi^{4/2}}{\Gamma(\frac{6}{2})}\left[\sqrt{\pi}\frac{\Gamma(\frac{6}{2})}{\Gamma(\frac{7}{2})}\right] = \frac{\pi^{5/2}}{\Gamma(\frac{7}{2})}.
\end{aligned}$$

Clearly,

$$C_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} = \frac{\pi^{d/2}}{(\frac{d}{2})!}.$$