Thermodynamics of the fermion gas

a. The grand canonical partition function is

$$\begin{aligned}\Xi &= \prod_r (1+e^{-\beta(\epsilon_r-\mu)})\\ \ln\Xi &= \sum_r \ln(1+e^{-\beta(\epsilon_r-\mu)}) = \int_0^\infty d\mathcal{E}\, G(\mathcal{E})\ln(1+e^{-\beta(\mathcal{E}-\mu)}),\end{aligned}$$

but

$$\Pi = -pV = -k_BT\ln\Xi \quad \text{so} \quad p(T,\mu)V = k_BT\int_0^\infty G(\mathcal{E})\ln(1+e^{-\beta\mathcal{E}}e^{\beta\mu})\,d\mathcal{E}.$$

b. For spin-$\frac{1}{2}$ particles, $G(\mathcal{E}) = V\left[\frac{2m^3}{\pi^4\hbar^6}\right]^{1/2}\sqrt{\mathcal{E}}$. Thus

$$\begin{aligned}p(T,\mu) &= k_BT\left[\frac{2m^3}{\pi^4\hbar^6}\right]^{1/2}\int_0^\infty \sqrt{\mathcal{E}}\ln(1+e^{-\beta\mathcal{E}}e^{\beta\mu})\,d\mathcal{E} \qquad [\![\ldots\text{use the substitution } x=\beta\mathcal{E}\ldots]\!]\\ &= (k_BT)^{5/2}\left[\frac{2m^3}{\pi^4\hbar^6}\right]^{1/2}\int_0^\infty \sqrt{x}\ln(1+e^{-x}e^{\beta\mu})\,dx.\end{aligned}$$

c. To evaluate

$$\int_0^\infty \sqrt{x}\ln(1+e^{-x}e^{\beta\mu})\,dx$$

integrate by parts using

$$\begin{aligned}du &= \sqrt{x}\,dx \qquad v = \ln(1+e^{-x}e^{\beta\mu})\\ u &= \tfrac{2}{3}x^{3/2} \qquad dv = \frac{dx}{1+e^{-x}e^{\beta\mu}}(-e^{-x}e^{\beta\mu}) = -\frac{dx}{e^xe^{-\beta\mu}+1}\end{aligned}$$

and find

$$\int_0^\infty \sqrt{x}\ln(1+e^{-x}e^{\beta\mu})\,dx = \left[\tfrac{2}{3}x^{3/2}\ln(1+e^{-x}e^{\beta\mu})\right]_0^\infty + \tfrac{2}{3}\int_0^\infty \frac{x^{3/2}}{e^xe^{-\beta\mu}+1}\,dx.$$

As $x\to\infty$, $x^{3/2}\ln(1+e^{-x}e^{\beta\mu}) \to x^{3/2}e^{-x}e^{\beta\mu} \to 0$, so the term in square brackets vanishes and we have

$$p(T,\mu) = \tfrac{2}{3}(k_BT)^{5/2}\left[\frac{2m^3}{\pi^4\hbar^6}\right]^{1/2}\int_0^\infty \frac{x^{3/2}}{e^xe^{-\beta\mu}+1}\,dx.$$

d.

$$\begin{aligned}E(T,V,\mu) &= \int_0^\infty G(\mathcal{E})f(\mathcal{E})\mathcal{E}\,d\mathcal{E}\\ &= V\left[\frac{2m^3}{\pi^4\hbar^6}\right]^{1/2}\int_0^\infty \sqrt{\mathcal{E}}\frac{1}{e^{\beta(\mathcal{E}-\mu)}+1}\mathcal{E}\,d\mathcal{E}\\ &= V(k_BT)^{5/2}\left[\frac{2m^3}{\pi^4\hbar^6}\right]^{1/2}\int_0^\infty \frac{x^{3/2}}{e^xe^{-\beta\mu}+1}\,dx\end{aligned}$$