Thermodynamics of the Bose condensate

For $T < T_0$, $\mu(T) = 0$, whence $b(\mathcal{E}) = \dfrac{1}{e^{\beta\mathcal{E}} - 1}$.

There is no energy in the "macroscopically populated" ground level ($\epsilon_1 = 0$), so

$$\begin{aligned}
E &= \int_{0+}^{\infty} G(\mathcal{E}) b(\mathcal{E}) \mathcal{E}\, d\mathcal{E} \\
&= V \left[\frac{m^3}{2\pi^4\hbar^6}\right]^{1/2} \int_0^\infty \sqrt{\mathcal{E}} \frac{1}{e^{\beta\mathcal{E}} - 1} \mathcal{E}\, d\mathcal{E} \qquad [\![\ldots \text{use the substitution } x = \beta\mathcal{E}, \text{ or } \mathcal{E} = k_B T x \ldots]\!] \\
&= V \left[\frac{m^3}{2\pi^4\hbar^6}\right]^{1/2} (k_B T)^{5/2} \int_0^\infty \frac{x^{3/2}}{e^x - 1}\, dx.
\end{aligned}$$

Even without evaluating the integral, we can define

$$I = \int_0^\infty \frac{x^{3/2}}{e^x - 1}\, dx$$

and know that this I is a dimensionless *number*. It isn't a function of T, or V, or anything else. Thus we can say

$$\begin{aligned}
E &= (k_B T)^{5/2} V \left[\frac{m^3}{2\pi^4\hbar^6}\right]^{1/2} I \\
C_V &= \left.\frac{\partial E}{\partial T}\right)_V \\
&= \tfrac{5}{2} k_B^{5/2} T^{3/2} V \left[\frac{m^3}{2\pi^4\hbar^6}\right]^{1/2} I \\
&= \frac{5}{2}\frac{E}{T} \\
S &= \int_0^T \frac{C_V}{T} dT \\
&= \tfrac{5}{2} k_B^{5/2} \left\{ \int_0^T \frac{T^{3/2}}{T} dT \right\} V \left[\frac{m^3}{2\pi^4\hbar^6}\right]^{1/2} I \\
&= \tfrac{5}{2} k_B^{5/2} \left\{ \frac{1}{3/2} T^{3/2} \right\} V \left[\frac{m^3}{2\pi^4\hbar^6}\right]^{1/2} I \\
&= \tfrac{5}{3} k_B^{5/2} T^{3/2} V \left[\frac{m^3}{2\pi^4\hbar^6}\right]^{1/2} I \\
&= \frac{5}{3}\frac{E}{T}.
\end{aligned}$$