Statistical Mechanics 2024

Model Solutions for First Exam

1. The coin toss

A single coin is tossed seven times.

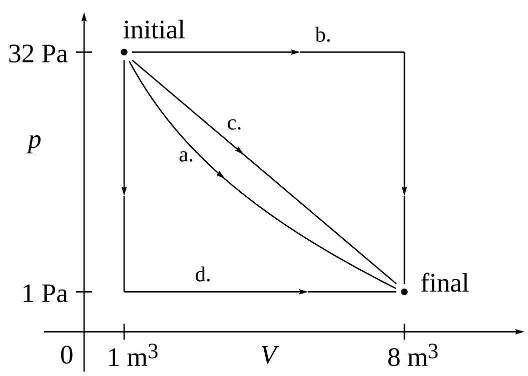
a. The probability of obtaining all heads is $1/2^7$.

b. The probability of obtaining alternating heads and tails is $2/2^7$. (One alternating pattern starts with heads, the other starts with tails.)

c. The probability of obtaining the pattern `THHTTHT` is $1/2^7$.

d. The probability of obtaining a pattern with one head and six tails is $7/2^7$. (There are seven such patterns.)

2. Fluid work



For a quasistatic change, the dissipative work is zero so

$$\text{work} = \text{configuration work} = -\int_{\text{initial}}^{\text{final}} p(V)\, dV.$$

Now along path (a)

$$p(V) = \frac{K}{V^\gamma} \quad \text{where} \quad K = p_i V_i^\gamma = p_f V_f^\gamma,$$

so the work along path (a) is

$$\text{Work} = -\int_{V_i}^{V_f} \frac{K}{V^\gamma}\, dV = \frac{K}{(\gamma-1)}\left[\frac{1}{V^{\gamma-1}}\right]_{V_i}^{V_f} = \frac{K}{(\gamma-1)}\left[\frac{1}{V_f^{\gamma-1}} - \frac{1}{V_i^{\gamma-1}}\right] = \frac{p_f V_f - p_i V_i}{\gamma - 1}.$$

Path (a) is quasistatic and adiabatic, so $Q = 0$ and $\Delta E = Q + W = W$. Plugging in numbers, $W = -36$ J, $Q = 0$, and $\Delta E = -36$ J. [Note that we never need to calculate any power V^γ.]

Now ΔE is the same for all paths. Thus for paths (b), (c), and (d) we can find W through "negative of area under the path" and Q through $Q = \Delta E - W$. The results are

path	W	Q
a	−36 J	0 J
b	−224 J	188 J
c	−115.5 J	79.5 J
d	−7 J	−29 J

3. Magnetic systems

For these systems (using $x = E/(NH)$)

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{df}{dx}\frac{\partial x}{\partial E} = f'(E/(NH))\left(\frac{1}{NH}\right)$$

$$\frac{M}{T} = \frac{\partial S}{\partial H} = \frac{df}{dx}\frac{\partial x}{\partial H} = f'(E/(NH))\left(-\frac{E}{NH^2}\right)$$

Thus

$$M = \frac{M/T}{1/T} = \frac{f'(E/(NH))\left(-\frac{E}{NH^2}\right)}{f'(E/(NH))\left(\frac{1}{NH}\right)} = -\frac{E}{H}.$$

This result might be familiar to you from an electricity and magnetism class in the form $E = -MH$.

4. From equation of state to entropy

Combining the two equations in the problem statement gives

$$\frac{p(E,V,N)}{T(E,V,N)} = \frac{\partial S(E,V,N)}{\partial V} = \frac{Nk_B}{V}.$$

If this were a single-variable problem then the right-hand equation would read

$$\frac{dS}{dV} = \frac{Nk_B}{V}$$

and you would immediately integrate this equation as

$$\begin{aligned} dS &= Nk_B \frac{dV}{V} \\ \int dS &= Nk_B \int \frac{dV}{V} \\ S &= Nk_B\left[\ln V + \text{constant}\right] = Nk_B \ln(V/V_0) \end{aligned}$$

where in the last step I have written the constant in the form $-\ln V_0$. I prefer this last form because it makes clear that V_0 is a constant with the dimensions of volume.

For the case where S is a function of three variables, the result is exactly the same except the integration over V is carried out at constant E and N, whence the "constant" V_0, although independent of V, might (and generally will) depend upon E and N. Thus

$$S(E,V,N) = k_B N \ln \frac{V}{V_0(E,N)}.$$

⟦You can check for yourself to see that, for the classical monatomic ideal gas, the function $V_0(E,N)$ is given through

$$\frac{1}{V_0(E,N)} = e^{5/2}\left(\frac{4\pi mE}{3h_0^2N^{5/3}}\right)^{3/2} .$$ ⟧