Schottky anomaly

$$\begin{aligned}
\zeta(T) &= \sum_{r=1}^{2} e^{-\beta\epsilon_r} = 1 + e^{-\beta\epsilon} \\
e^{\text{int}}(T) &= -\frac{\partial \ln \zeta}{\partial \beta} = -\frac{-\epsilon e^{-\beta\epsilon}}{1+e^{-\beta\epsilon}} = \epsilon \frac{1}{1+e^{\beta\epsilon}} \\
c_V^{\text{int}}(T) &= \frac{\partial e^{\text{int}}}{\partial T} = \frac{\partial \beta}{\partial T}\frac{\partial e^{\text{int}}}{\partial \beta} \\
&= \left(-\frac{1}{k_B T^2}\right) \epsilon \left(-\frac{\epsilon e^{\beta\epsilon}}{(1+e^{\beta\epsilon})^2}\right) \\
&= k_B \left(\frac{\epsilon}{k_B T}\right)^2 \frac{e^{-\epsilon/k_B T}}{(1+e^{-\epsilon/k_B T})^2}
\end{aligned}$$

High temperature behavior: $k_B T \gg \epsilon$; $\beta\epsilon \ll 1$; $e^{-\beta\epsilon} \approx 1$

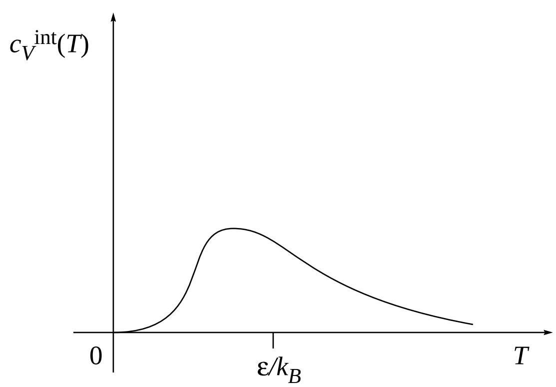
$$c_V^{\text{int}}(T) \approx \frac{k_B}{4}\left(\frac{\epsilon}{k_B T}\right)^2$$

Low temperature behavior: $k_B T \ll \epsilon$; $\beta\epsilon \gg 1$; $e^{-\beta\epsilon} \approx 0$

$$c_V^{\text{int}}(T) \approx k_B \left(\frac{\epsilon}{k_B T}\right)^2 e^{-\epsilon/k_B T}$$

This is a very slowly increasing function of T: It not only has zero slope at $T = 0$, but also zero curvature at $T = 0$, zero third derivative at $T = 0$, and in fact all derivatives of $c_V^{\text{int}}(T)$ are zero at $T = 0$:

$$\frac{\partial^n c_V^{\text{int}}}{\partial T^n}(0) = 0 \qquad \text{for } n = 0, 1, 2, \ldots$$



The maximum of this curve is in the vicinity of the characteristic temperature ϵ/k_B, although it would be an amazing coincidence if the maximum were exactly at that point. This curve is *not* symmetric. It falls off rapidly (exponentially) on the low-temperature side and slowly (algebraically) on the high-temperature side.