Parametric differentiation in quantum mechanics

Start with

$$\int_0^\pi \sin(ax)\sin(bx)\,dx = \frac{1}{2}\left[\frac{\sin[(a-b)\pi]}{a-b} - \frac{\sin[(a+b)\pi]}{a+b}\right] \quad a \neq \pm b. \tag{1}$$

Looking on the left side:

$$\frac{\partial^2}{\partial a^2}\left[\int_0^\pi (\sin\, ax)(\sin\, bx)\,dx\right] = \int_0^\pi \left(\frac{\partial^2}{\partial a^2}\sin\, ax\right)(\sin\, bx)\,dx = -\int_0^\pi (\sin\, ax)x^2(\sin\, bx)\,dx.$$

Looking on the right side:

$$\begin{aligned}
\frac{\partial}{\partial u}\left[\frac{\sin\, u\pi}{u}\right] &= \frac{u\pi\cos\, u\pi - \sin\, u\pi}{u^2} \\
\frac{\partial^2}{\partial u^2}\left[\frac{\sin\, u\pi}{u}\right] &= \frac{u^2[\pi\cos\, u\pi - u\pi^2\sin\, u\pi - \pi\cos\, u\pi] - 2u[u\pi\cos\, u\pi - \sin\, u\pi]}{u^4} \\
&= -\frac{\pi^2\sin\, u\pi}{u} - \frac{2\pi\cos\, u\pi}{u^2} + \frac{2\sin\, u\pi}{u^3}.
\end{aligned}$$

Thus, the second derivative of equation (1) with respect to a becomes

$$\begin{aligned}
\int_0^\pi (\sin\, ax)x^2(\sin\, bx)\,dx &= -\frac{1}{2}\frac{\partial^2}{\partial a^2}\left[\frac{\sin(a-b)\pi}{a-b} - \frac{\sin(a+b)\pi}{a+b}\right] \\
&= +\frac{1}{2}\left[\frac{\pi^2\sin(a-b)\pi}{a-b} + \frac{2\pi\cos(a-b)\pi}{(a-b)^2} - \frac{2\sin(a-b)\pi}{(a-b)^3}\right] \\
&\quad -\frac{1}{2}\left[\frac{\pi^2\sin(a+b)\pi}{a+b} + \frac{2\pi\cos(a+b)\pi}{(a+b)^2} - \frac{2\sin(a+b)\pi}{(a+b)^3}\right].
\end{aligned}$$

Recall that for N an integer, $\sin(N\pi) = 0$ and $\cos(N\pi) = (-1)^N$. This tells us that for n and m integers, (with $n \neq m$)

$$\begin{aligned}
\langle n|x^2|m\rangle &= \frac{2}{\pi}\int_0^\pi (\sin\, nx)x^2(\sin\, mx)\,dx \\
&= 2\left[\frac{(-1)^{n-m}}{(n-m)^2} - \frac{(-1)^{n+m}}{(n+m)^2}\right] \\
&= 2(-1)^{n+m}\left[\frac{1}{(n-m)^2} - \frac{1}{(n+m)^2}\right] \\
&= (-1)^{n+m}\left[\frac{8\,nm}{(n^2-m^2)^2}\right].
\end{aligned}$$