Number fluctuations in the grand canonical ensemble

$$\Xi(T,V,\mu) = \sum_{N=0}^{\infty} e^{\beta\mu N} Z(T,V,N)$$

$$\left.\frac{\partial \ln \Xi}{\partial \mu}\right)_{T,V} = \frac{1}{\Xi}\frac{\partial \Xi}{\partial \mu} = \beta \frac{\displaystyle\sum_{N=0}^{\infty} N e^{\beta\mu N} Z_N}{\Xi} = \beta\langle N\rangle$$

$$\begin{aligned}
\frac{\partial^2 \ln \Xi}{\partial \mu^2} &= \frac{\partial}{\partial \mu}\left(\frac{1}{\Xi}\frac{\partial \Xi}{\partial \mu}\right) \\
&= -\left(\frac{1}{\Xi}\frac{\partial \Xi}{\partial \mu}\right)^2 + \frac{1}{\Xi}\frac{\partial^2 \Xi}{\partial \mu^2} \\
&= -\beta^2\langle N\rangle^2 + \beta^2\langle N^2\rangle \\
&= \beta^2(\Delta N)^2
\end{aligned}$$

Multiply both sides by $-k_BT$

$$\frac{\partial^2 \Pi}{\partial \mu^2} = -\frac{(\Delta N)^2}{k_BT} \quad \text{whence} \quad \left.\frac{\partial^2 (pV)}{\partial \mu^2}\right)_{T,V} = \frac{(\Delta N)^2}{k_BT} \quad \text{whence} \quad V\left.\frac{\partial^2 p}{\partial \mu^2}\right)_{T} = \frac{(\Delta N)^2}{k_BT}$$

But from problem 3.31, "Isothermal compressibility,"

$$\left.\frac{\partial^2 p}{\partial \mu^2}\right)_{T} = \rho^2\kappa_T = \frac{N^2}{V^2}\kappa_T,$$

so

$$(\Delta N)^2 = k_BT\frac{N^2}{V}\kappa_T$$

$$\frac{\Delta N}{N} = \sqrt{k_BT\frac{\kappa_T}{V}}$$