The logarithm

Start with

$$f(xy) = f(x) + f(y). \tag{1}$$

Differentiate with respect to y:

$$\begin{aligned}
\frac{\partial f(xy)}{\partial y} &= 0 + f'(y) \\
\frac{\partial f(xy)}{\partial (xy)} \frac{\partial (xy)}{\partial y} &= f'(y) \\
f'(xy)x &= f'(y) \qquad (2)
\end{aligned}$$

Set $y = 1$:

$$f'(x)x = f'(1) \equiv k. \tag{3}$$

Then

$$\begin{aligned}
f'(x)x &= k \\
f'(x) &= \frac{k}{x} \\
f(x) &= k \ln(x/x_0)
\end{aligned}$$

where x_0 is a constant of integration. (The argument in this paragraph is due to Naiyuan Zhang '18.)

What is x_0? Take $y = 1$ in equation (1), giving

$$f(x) = f(x) + f(1).$$

Hence $f(1) = 0$, whence $x_0 = 1$, and

$$f(x) = k \ln(x).$$

(The argument of this paragraph is due to Bryce Denny '98.)