Isothermal compressibility

a. The isothermal compressibility is

$$\kappa_T = -\frac{1}{V}\left.\frac{\partial V}{\partial p}\right)_{T,N}, \tag{1}$$

but $\rho = N/V$, so $V = N/\rho$, so

$$\kappa_T = -\frac{1}{[N/\rho]}\left.\frac{\partial [N/\rho]}{\partial p}\right)_{T,N} = -\frac{\rho}{N} N \left.\frac{\partial [1/\rho]}{\partial p}\right)_{T}. \tag{2}$$

In the last step, we changed from a derivative at constant T and N to a derivative at constant T alone because in the right-most derivative all the quantities are intensive. The variable N *is* constant during this differentiation, just as the elevation of Mount Everest is constant during this differentiation, but neither are listed because neither are relevant (neither appear in the list of variables). Continuing:

$$\kappa_T = -\rho \left.\frac{\partial [1/\rho]}{\partial p}\right)_{T} = \frac{1}{\rho}\left.\frac{\partial \rho}{\partial p}\right)_{T}. \tag{3}$$

Now, from the single-variable chain rule

$$\left.\frac{\partial \rho}{\partial p}\right)_{T} = \left.\frac{\partial \rho}{\partial \mu}\right)_{T}\left.\frac{\partial \mu}{\partial p}\right)_{T}. \tag{4}$$

But, because $dp = \mathcal{S}\, dT + \rho\, d\mu$,

$$\rho = \left.\frac{\partial p}{\partial \mu}\right)_{T}. \tag{5}$$

Thus

$$\kappa_T = \frac{1}{\rho}\left.\frac{\partial \rho}{\partial \mu}\right)_{T}\frac{1}{\rho} = \frac{1}{\rho^2}\left.\frac{\partial \rho}{\partial \mu}\right)_{T}. \tag{6}$$

Furthermore, using equation (5), we have

$$\kappa_T = \frac{1}{\rho^2}\left.\frac{\partial^2 p}{\partial \mu^2}\right)_{T}. \tag{7}$$

b. What does this result tell you about the relation between density and chemical potential?

$$\left.\frac{\partial \rho}{\partial \mu}\right)_{T} = \rho^2 \kappa_T > 0, \tag{8}$$

so at any given temperature, the density increases with increasing chemical potential.

c. Returning to equation (6), we can reverse the process described under equation (2) and write

$$\left.\frac{\partial \rho}{\partial \mu}\right)_{T} = \left.\frac{\partial [N/V]}{\partial \mu}\right)_{T} = [1/V]\left.\frac{\partial N}{\partial \mu}\right)_{T,V}, \tag{9}$$

or, alternately,

$$\left.\frac{\partial \rho}{\partial \mu}\right)_{T} = \left.\frac{\partial [N/V]}{\partial \mu}\right)_{T} = N\left.\frac{\partial [1/V]}{\partial \mu}\right)_{T,N} = -\frac{N}{V^2}\left.\frac{\partial V}{\partial \mu}\right)_{T,N}. \tag{10}$$

Plugging both of these results into equation (6) gives

$$\kappa_T = \frac{V}{N^2}\left.\frac{\partial N}{\partial \mu}\right)_{T,V} = -\frac{1}{N}\left.\frac{\partial V}{\partial \mu}\right)_{T,N}. \tag{11}$$