Isothermal compressibility of quantal ideal gases

a.

$$\kappa_T = \frac{V}{N^2}\left.\frac{\partial N}{\partial \mu}\right)_{T,V}$$

$$N = \sum_r \langle n_r \rangle = \sum_r \frac{1}{e^{\beta(\epsilon_r-\mu)} \pm 1}$$

$$\left.\frac{\partial N}{\partial \mu}\right)_{T,V} = \sum_r -\frac{-\beta e^{\beta(\epsilon_r-\mu)}}{(e^{\beta(\epsilon_r-\mu)} \pm 1)^2} = \beta \sum_r \frac{e^{\beta(\epsilon_r-\mu)}}{(e^{\beta(\epsilon_r-\mu)} \pm 1)^2}$$

$$\langle n_r \rangle = \frac{1}{e^{\beta(\epsilon_r-\mu)} \pm 1} \quad \text{so} \quad e^{\beta(\epsilon_r-\mu)} = \frac{1}{\langle n_r \rangle} \mp 1$$

$$\left.\frac{\partial N}{\partial \mu}\right)_{T,V} = \beta \sum_r \langle n_r \rangle^2 \left(\frac{1}{\langle n_r \rangle} \mp 1\right) = \beta \sum_r \left(\langle n_r \rangle \mp \langle n_r \rangle^2\right) = \beta \left(N \mp \sum_r \langle n_r \rangle^2\right)$$

$$\kappa_T = \frac{1}{\rho N}\left.\frac{\partial N}{\partial \mu}\right)_{T,V} = \frac{\beta}{\rho N}\left(N \mp \sum_r \langle n_r \rangle^2\right) = \frac{1}{\rho k_B T}\left[1 \mp \frac{\sum_r \langle n_r \rangle^2}{\sum_r \langle n_r \rangle}\right]$$

b. Classical: $pV = Nk_BT$; $\kappa_T = \frac{1}{p} = \frac{1}{\rho k_B T}$

c.

$$\kappa_T = \frac{1}{\rho k_B T}\left[\frac{\sum_r \langle n_r \rangle - \langle n_r \rangle^2}{N}\right] = \frac{1}{\rho N k_B T}\sum_r \langle n_r \rangle(1 - \langle n_r \rangle)$$

For fermions, $0 \le \langle n_r \rangle \le 1$, so $(1 - \langle n_r \rangle)$ is non-negative.

d. The sequence is

$$\text{hardest} \quad \kappa_T^{\mathrm{FD}} < \kappa_T^{\mathrm{MB}} < \kappa_T^{\mathrm{BE}} \quad \text{softest}$$

. . . as expected from the effective repulsion of identical fermions and effective attraction of identical bosons. "Identical fermions spread apart, identical bosons huddle together."