Ideal paramagnet, take three

From the problem "Entropy of a spin system" ...

$$S(E,H,N) = -k_B N \left\{ \left[\tfrac{1}{2}(1+u)\right] \ln\left[\tfrac{1}{2}(1+u)\right] + \left[\tfrac{1}{2}(1-u)\right] \ln\left[\tfrac{1}{2}(1-u)\right] \right\} \equiv -k_B N f(u)$$

where

$$u = \frac{E}{NmH}.$$

And from the definitions

$$-\frac{\mu}{T} = \frac{\partial S}{\partial N} \quad \text{and} \quad \frac{1}{T} = \frac{\partial S}{\partial E}$$

we have

$$\mu(E,H,N) = -\frac{\partial S}{\partial N} \bigg/ \frac{\partial S}{\partial E}.$$

Now

$$\frac{\partial S}{\partial N} = -k_B f(u) - k_B N f'(u) \left(-\frac{E}{N^2 mH} \right) = -k_B f(u) + k_B u f'(u)$$

$$\frac{\partial S}{\partial E} = -k_B N f'(u) \left(\frac{1}{NmH} \right) = -\frac{k_B}{mH} f'(u)$$

so

$$\mu = -mH \frac{f(u) - u f'(u)}{f'(u)}.$$

Meanwhile

$$\begin{aligned} f(u) &= \left[\tfrac{1}{2}(1+u)\right] \ln\left[\tfrac{1}{2}(1+u)\right] + \left[\tfrac{1}{2}(1-u)\right] \ln\left[\tfrac{1}{2}(1-u)\right] \\ &= \left[\tfrac{1}{2}(1+u)\right] \ln\tfrac{1}{2} + \left[\tfrac{1}{2}(1+u)\right] \ln(1+u) + \left[\tfrac{1}{2}(1-u)\right] \ln\tfrac{1}{2} + \left[\tfrac{1}{2}(1-u)\right] \ln(1-u) \\ &= \ln\tfrac{1}{2} + \left[\tfrac{1}{2}(1+u)\right] \ln(1+u) + \left[\tfrac{1}{2}(1-u)\right] \ln(1-u) \\ &= -\ln 2 + \left[\tfrac{1}{2}(1+u)\right] \ln(1+u) + \left[\tfrac{1}{2}(1-u)\right] \ln(1-u). \end{aligned}$$

So

$$\begin{aligned} f'(u) &= \left[\tfrac{1}{2}\right] \ln(1+u) + \tfrac{1}{2} + \left[-\tfrac{1}{2}\right] \ln(1-u) - \tfrac{1}{2} \\ &= \tfrac{1}{2}\left[\ln(1+u) - \ln(1-u)\right] \end{aligned}$$

and

$$f(u) - u f'(u) = -\ln 2 + \tfrac{1}{2}\ln(1+u) + \tfrac{1}{2}\ln(1-u)$$

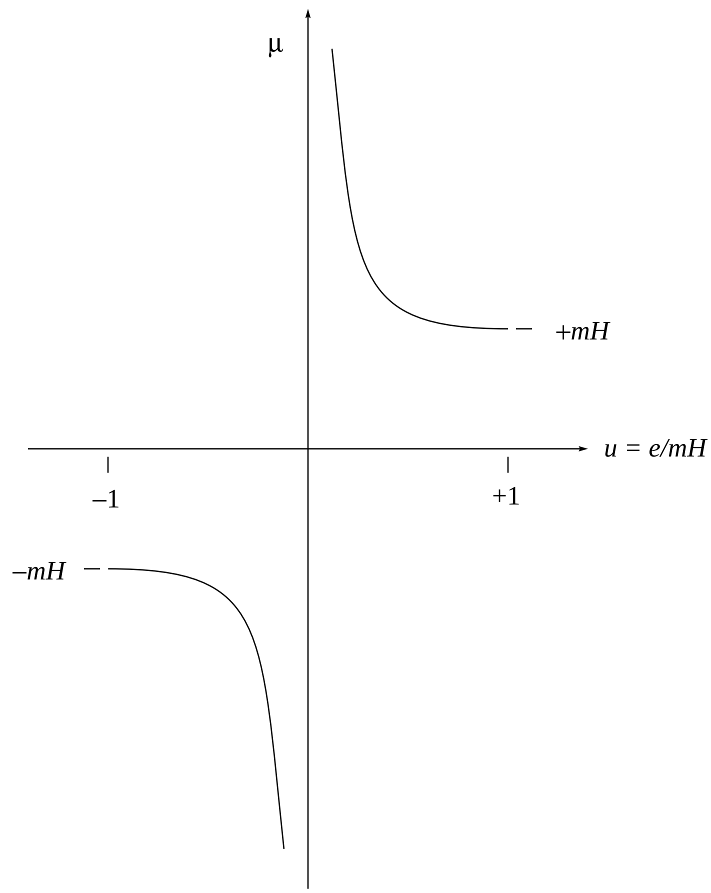
In conclusion

$$\mu = mH \left[\frac{2\ln 2 - \ln(1+u) - \ln(1-u)}{\ln(1+u) - \ln(1-u)} \right].$$

(Sketch on next page.)

Here's a sketch of

$$\mu = mH \left[\frac{2\ln 2 - \ln(1+u) - \ln(1-u)}{\ln(1+u) - \ln(1-u)} \right]. \tag{1}$$



You can express this result in other ways, for example as

$$\mu = mH \left[\frac{\ln\left\{4/(1-u^2)\right\}}{\ln\left\{(1+u)/(1-u)\right\}} \right] \tag{2}$$

or as

$$\mu = mH \left[\frac{\ln\left\{(1-u^2)/4\right\}}{\ln\left\{(1-u)/(1+u)\right\}} \right], \tag{3}$$

but I find these other expressions less insightful than the expression (1) that we first derived. For example, the function is odd under reflection, that is $\mu(u) = -\mu(-u)$: this fact is transparent in expression (1) and obscure in expressions (2) and (3).