Fluctuation-susceptibility relation for a magnetic system

$$Z(T,H) = \sum_{\text{states}} e^{-\beta\mathcal{H}_0} e^{\beta m H \sum_i s_i} = \sum_{\text{states}} e^{-\beta\mathcal{H}_0} e^{\beta H \mathcal{M}}.$$

In the rest of this problem, the sum sign indicates a sum over states:

$$\sum \quad \text{means} \quad \sum_{\text{states}}.$$

Now

$$\langle\mathcal{M}\rangle = \frac{\sum \mathcal{M} e^{-\beta\mathcal{H}_0} e^{\beta H\mathcal{M}}}{Z}$$

and

$$\langle\mathcal{M}^2\rangle = \frac{\sum \mathcal{M}^2 e^{-\beta\mathcal{H}_0} e^{\beta H\mathcal{M}}}{Z}.$$

Thus

$$\begin{aligned}
\chi_H &= \frac{\partial\langle\mathcal{M}\rangle}{\partial H} \\
&= \frac{Z\beta \sum \mathcal{M}^2 e^{-\beta\mathcal{H}_0} e^{\beta H\mathcal{M}} - \frac{\partial Z}{\partial H} \sum \mathcal{M} e^{-\beta\mathcal{H}_0} e^{\beta H\mathcal{M}}}{Z^2} \\
&= \beta\langle\mathcal{M}^2\rangle - \frac{\beta \sum \mathcal{M} e^{-\beta\mathcal{H}_0} e^{\beta H\mathcal{M}} \sum \mathcal{M} e^{-\beta\mathcal{H}_0} e^{\beta H\mathcal{M}}}{Z^2} \\
&= \beta\langle\mathcal{M}^2\rangle - \beta\langle\mathcal{M}\rangle^2 \\
&= \beta\Delta M^2.
\end{aligned}$$

Therefore

$$\Delta M = \sqrt{k_B T \chi_H}.$$