

Name: _____

Date: _____ Start: _____ am/pm Finish: _____ am/pm

Statistical Mechanics

Final Examination: due 9:00 PM on Thursday, 16 May 2019

- Do not open this packet until you are ready to work the problems.
- All work on the exam must be done in an unbroken two hour period that begins when you open the exam. If you work the problems on scrap paper and then make a fine copy to submit, the copying as well as the working must be completed within the two hour limit.
- Submit your solutions to me in Wright 215 by the due time given above. Until that instant you may not discuss the exam with anyone other than me. If you find a problem statement unclear, you may visit or call me (extension 5-8183, cell phone 440-281-1348, between 7:00 AM and 9:00 PM only).
- You may consult any written or on-line materials including your notes, but no collaboration is permitted.
- Solutions must be complete enough that I can follow the logic.
- Write your answers on $8\frac{1}{2}$ " \times 11" pages and staple them to this question page before submitting them. Attach all scrap paper that you used as well.
- Adhere to the requirements of the honor system. Sign the honor pledge below.
- Enjoy your summer break and, for that matter, enjoy the rest of your life. Teaching this class has been a very pleasant adventure for me.

I affirm that I have adhered to the Honor Code in this assignment.

	points earned	maximum
problem 1		10
problem 2		10
problem 3		10
problem 4		10
total		40

1 Change of chemical potential with temperature

Prove that

$$\left(\frac{\partial \mu}{\partial T}\right)_{p,N} = -\frac{S}{N} \quad (1)$$

for a fluid system.

2 Generalized equipartition theorem and the ultra-relativistic gas

- a. Suppose the Hamiltonian $H(\Gamma)$ decouples into two pieces

$$H(\Gamma) = a|p|^n + H_2(\Gamma_2) \quad (2)$$

where p is some phase space variable that may take on values from $-\infty$ to $+\infty$, and where Γ_2 represents all the phase space variables except for p . (Note that the absolute value $|p|$ is needed in order to avoid, for example, taking the square root of a negative number in the case $n = 1/2$.) Show that, in classical statistical mechanics, the mean contribution to the energy due to that single variable is

$$\langle a|p|^n \rangle = \frac{1}{n} k_B T. \quad (3)$$

- b. In special relativity, the energy of a free (i.e. non-interacting) particle is given by

$$\sqrt{(mc^2)^2 + (pc)^2}, \quad (4)$$

where c is the speed of light. As you know, when $v \ll c$ this gives the non-relativistic kinetic energy $\text{KE} \approx mc^2 + p^2/2m$. In the “ultra-relativistic” limit, where v is close to c , the energy is approximately pc . What is the heat capacity of a gas of non-interacting ultra-relativistic particles?

- c. Estimate the crossover temperature between the non-relativistic and ultra-relativistic regimes.

3 Fermion gas in two dimensions

Consider a gas of free, independent, spin- $\frac{1}{2}$ fermions in two dimensions. The gas is contained within an area (or two dimensional volume) of A .

- a. What is the density of one-particle levels in k -space?
b. How does the Fermi energy \mathcal{E}_F depend upon the density N/A ?
c. Use $\sum_r \langle n_r \rangle = N$ to show that

$$\mu + k_B T \ln(1 + e^{-\mu/k_B T}) = \mathcal{E}_F. \quad (5)$$

- d. Does the chemical potential μ increase or decrease with temperature?

~OVER~

4 Polymers

A primitive model for a polymer is a random walk on a simple cubic lattice. A random walk consists of n steps (or “links”) starting at (“anchored to”) the origin. (In this model a polymer unit can step back onto a lattice site already occupied by a different polymer unit. This unrealistic feature is corrected in a more sophisticated model, the so-called “self avoiding walk.”)

- a. Show that the number of distinct walks consisting of n links is $N_n = 6^n$. Does this formula hold when $n = 0$?

For many purposes it is valuable to consider the ensemble of *all* random walks, regardless of their size. In this ensemble there is a “size control parameter” α such that the probability of finding a walk x consisting of $n(x)$ links is proportional to $e^{-\alpha n(x)}$. (Thus *longer* walks are *less* probable in this ensemble, but there are more of them.) The partition function associated with this model is

$$\Xi(\alpha) = \sum_{\text{walks } x} e^{-\alpha n(x)}. \quad (6)$$

- b. Show that the mean walk size in this ensemble is a function of α given through

$$\langle n \rangle = -\frac{\partial \ln \Xi(\alpha)}{\partial \alpha}. \quad (7)$$

- c. Show that

$$\Xi(\alpha) = \frac{1}{1 - 6e^{-\alpha}} \quad \text{and that} \quad \langle n \rangle = \frac{6}{e^{\alpha} - 6}. \quad (8)$$

Clue: The geometric series sums to

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1 - x} \quad \text{when} \quad |x| < 1.$$

- d. What is the smallest possible value of the control parameter α ? Does large α correspond to long polymers or short polymers?
- e. Show that the dispersion in n is given through

$$(\Delta n)^2 = \frac{\partial^2 \ln \Xi}{\partial \alpha^2}, \quad (9)$$

whence

$$\frac{\Delta n}{\langle n \rangle} = \sqrt{\frac{e^{\alpha}}{6}} = \sqrt{\frac{1}{\langle n \rangle} + 1}. \quad (10)$$

Thus the relative dispersion decreases for longer polymers.