Classical monatomic ideal gas in the grand canonical ensemble

a. In general

$$\Xi(T, V, \mu) = \sum_{N=0}^{\infty} e^{\beta\mu N} Z(\beta, V, N).$$

(Note that $\mu < 0$ — otherwise this sum would usually diverge.) For the classical monatomic ideal gas

$$Z(T, V, N) = \frac{1}{N!}\left[\frac{V}{\lambda^3(T)}\right]^N,$$

so

$$\Xi(T, V, \mu) = \sum_{N=0}^{\infty} \frac{1}{N!}\left[e^{\beta\mu}\frac{V}{\lambda^3(T)}\right]^N.$$

But

$$e^z = \sum_{N=0}^{\infty} \frac{1}{N!} z^N,$$

so

$$\Xi(T, V, \mu) = \exp\left\{\frac{e^{\beta\mu}V}{\lambda^3(T)}\right\}. \tag{1}$$

b. Thermodynamics tells us that for any pure fluid (not just the ideal gas)

$$N(T, V, \mu) = -\left.\frac{\partial \Pi}{\partial \mu}\right)_{T,V} = \left.\frac{\partial p(T, \mu)V}{\partial \mu}\right)_{T,V}$$

(see, for example, appendix J). So the connection between thermodynamics and statistical mechanics for the grand canonical ensemble, namely

$$p(T, \mu)V = k_B T \ln \Xi(T, V, \mu), \tag{2}$$

implies that

$$N(T, V, \mu) = k_B T \left.\frac{\partial \ln \Xi}{\partial \mu}\right)_{T,V}. \tag{3}$$

c. For the classical monatomic ideal gas, equation (1) implies that

$$\ln \Xi(T, V, \mu) = e^{\beta\mu}\frac{V}{\lambda^3(T)}. \tag{4}$$

So equation (3) says that

$$N(T, V, \mu) = k_B T\left[\beta e^{\beta\mu}\frac{V}{\lambda^3(T)}\right] = e^{\beta\mu}\frac{V}{\lambda^3(T)}.$$

But looking back at (2) and (4) shows that

$$N(T, V, \mu) = \frac{p(T, \mu)V}{k_B T}$$

or

$$pV = N k_B T.$$