From the “three principles” to the Lorentz transformation and back

by Dan Styer © 18 September 2014

The three principles: My book *Relativity for the Questioning Mind* summarizes special relativity (for events on the x axis) as the “three principles”:

Time dilation	A moving clock ticks slowly.	$T = \dfrac{T_0}{\sqrt{1-(V/c)^2}}$

T_0 is the time ticked off by a single moving clock (which is also the time elapsed in that clock’s own frame).
T is the (longer) time elapsed in the frame in which that clock moves at speed V.

Length contraction	A moving rod is short.	$L = L_0\sqrt{1-(V/c)^2}$

L_0 is the length of a rod in that rod’s own frame (its “rest length”).
L is the (shorter) length of that rod in the frame in which that rod moves at speed V.

Relativity of synchronization	A moving pair of clocks isn’t synchronized.	Rear clock set ahead by L_0V/c^2.
Also called: Relativity of simultaneity	If two events are simultaneous in one frame, then in another frame the rear event happens first.	

The Lorentz transformation: It is far more traditional to summarize special relativity through the Lorentz transformation:

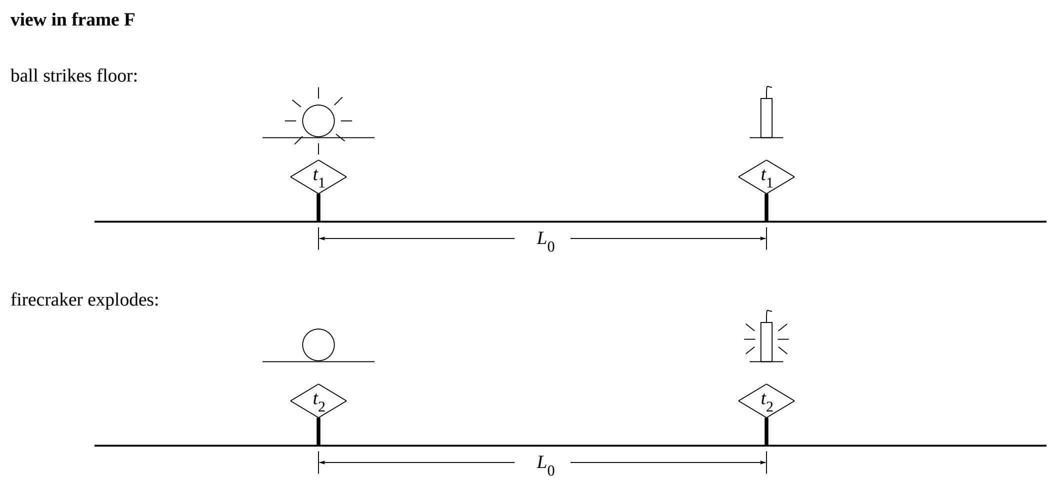
$$
\begin{aligned}
x' &= \frac{x - Vt}{\sqrt{1-(V/c)^2}}\\
t' &= \frac{t - Vx/c^2}{\sqrt{1-(V/c)^2}}
\end{aligned}
$$

This document will prove that the three principles and the Lorentz transformation are equivalent. That is, starting from the three principles you can derive the Lorentz transformation, and starting from the Lorentz transformation you can derive the three principles.

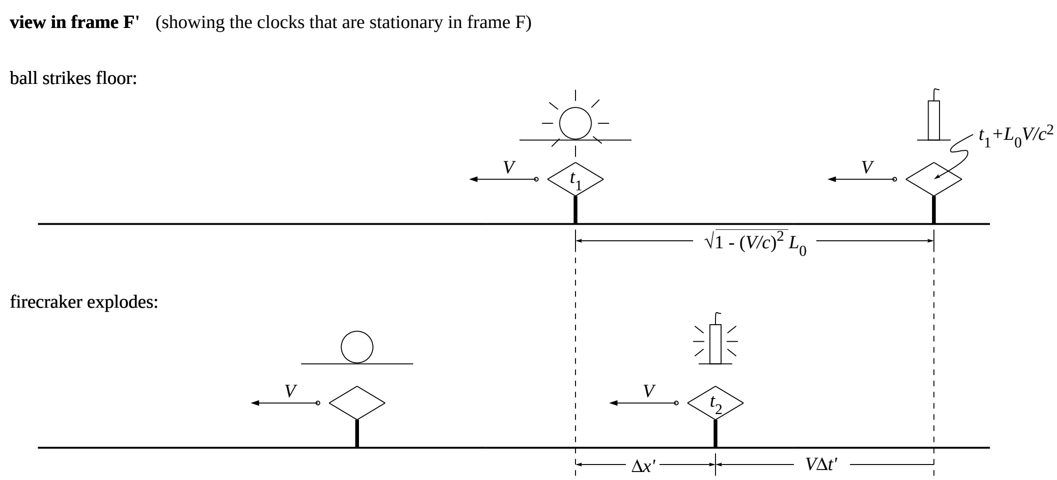
From the three to the LT

Two events occur: for example, at one time and place a falling ball strikes the floor, and then at some time and place a firecracker explodes. (The ball strike does not cause the firecracker to explode. These are just two events that I dreamed up. The figures on the next page show the ball striking the floor and then resting where it struck; they show the firecracker at rest on the floor before exploding.) In frame F these events are separated by distance Δx and time Δt. In frame F$'$ they are separated by distance $\Delta x'$ and time $\Delta t'$. How are these coordinates related?

In frame F, the two events are separated by space $\Delta x = L_0$ and by time $\Delta t = t_2 - t_1$:



In frame F′, the two events are separated by space $\Delta x'$ and time $\Delta t'$:



From the figure, $\Delta x' + V\Delta t' = \Delta x\sqrt{1-(V/c)^2}$ or

$$\Delta x = \frac{\Delta x' + V\Delta t'}{\sqrt{1-(V/c)^2}}. \tag{1}$$

How much time $\Delta t'$ elapses between the two events? The right hand clock (a single moving clock) ticks off a time

$$T_0 = t_2 - (t_1 + \Delta x V/c^2) = \Delta t - \Delta x V/c^2.$$

But this moving clock ticks slowly: the time elapsed is not the time ticked off by the clock, it is the larger time

$$\Delta t' = \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}}. \tag{2}$$

Solving equations (1) and (2) simultaneously for $\Delta x'$ in terms of Δx and Δt gives

$$\begin{aligned} \Delta x' &= \frac{\Delta x - V\Delta t}{\sqrt{1-(V/c)^2}} \\ \Delta t' &= \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}}. \end{aligned}$$

These equations transform the coordinates for any pair of events on the x axis. Suppose the first event is "the origin of frame F′ passes over the origin of frame F, an event that serves as the origin of time (i.e. $t = t' = 0$)". In this case the first event has coordinates $x_1 = 0$, $t_1 = 0$ in frame F and coordinates $x'_1 = 0$, $t'_1 = 0$ in frame F′. Thus the coordinates of the second event (now called simply x, t, and x', t') transform as

$$\begin{aligned} x' &= \frac{x - Vt}{\sqrt{1-(V/c)^2}} \\ t' &= \frac{t - Vx/c^2}{\sqrt{1-(V/c)^2}}, \end{aligned}$$

the Lorentz transformation!

From the LT to the three

Start with Lorentz transformation

$$\begin{aligned} x' &= \frac{x - Vt}{\sqrt{1-(V/c)^2}} \\ t' &= \frac{t - Vx/c^2}{\sqrt{1-(V/c)^2}}. \end{aligned}$$

If there are two events, then they are separated by

$$\Delta x' = \frac{\Delta x - V\Delta t}{\sqrt{1-(V/c)^2}} \tag{3}$$

$$\Delta t' = \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}}. \tag{4}$$

Further, frame F′ is just as good an inertial frame as frame F, so we can transform from coordinates in F′ to coordinates in F simply by replacing every V with a $-V$:

$$\Delta x = \frac{\Delta x' + V\Delta t'}{\sqrt{1-(V/c)^2}} \tag{5}$$

$$\Delta t = \frac{\Delta t' + V\Delta x'/c^2}{\sqrt{1-(V/c)^2}}. \tag{6}$$

To derive **time dilation**, think about what time dilation means: The single moving clock ticks twice — two events. The clock is stationary in frame F′, so these two ticks are separated by $\Delta x' = 0$ and $\Delta t' = T_0$. In frame F, the time elapsed T is given by equation (6), so

$$T = \frac{T_0}{\sqrt{1-(V/c)^2}}.$$

To derive **length contraction**, think about measuring the length of a rod moving in the laboratory: arrange for two events, simultaneous in the laboratory frame ($\Delta t = 0$), to occur at the two ends of the moving rod. These events are separated by length $L = \Delta x$ in the laboratory frame. In the rod's frame F′, the two events are not simultaneous, but they don't need to be: the distance between them is $\Delta x' = L_0$, regardless of $\Delta t'$, because the rod is at rest. Equation (3) gives the relationship

$$L_0 = \frac{L}{\sqrt{1-(V/c)^2}}$$

which is usually written

$$L = L_0\sqrt{1-(V/c)^2}\,.$$

The third principle is **relativity of synchronization**. A pair of moving clocks ticks simultaneously ($\Delta t' = 0$) in their own frame (F′), and the distance between them in that frame is $L_0 = \Delta x'$. In the laboratory frame the ticks are not simultaneous: according to equation (6), those two ticks are separated by a time

$$\frac{VL_0/c^2}{\sqrt{1-(V/c)^2}}.$$

But just because those two ticks are separated by this much time doesn't mean that this is the difference in the times announced by these two clocks: The two clocks are ticking slowly (time dilation) so the difference in time announced is the smaller time $\sqrt{1-(V/c)^2}$ times the above, namely

$$\frac{L_0 V}{c^2}.$$