Pauli matrix algebra

a.

$$\begin{aligned}
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= z_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix}
\end{aligned}$$

whence

$$\begin{aligned}
z_0 &= \tfrac{1}{2}(a_{11} + a_{22}) \\
z_1 &= \tfrac{1}{2}(a_{12} + a_{21}) \\
z_2 &= \tfrac{1}{2}i(a_{12} - a_{21}) \\
z_3 &= \tfrac{1}{2}(a_{11} - a_{22})
\end{aligned}$$

b. Just plug and chug! You need to work out these nine matrix products:

$$\begin{aligned}
\sigma_1\sigma_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
\sigma_1\sigma_2 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
\sigma_1\sigma_3 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
\sigma_2\sigma_1 &= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
\sigma_2\sigma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\sigma_2\sigma_3 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
\sigma_3\sigma_1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
\sigma_3\sigma_2 &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
\sigma_3\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Once you work them out you'll see that all the claims I made are correct.

c.

$$
\begin{aligned}
& (c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3)^2 \\
= \;& (c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3)(c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3) \\
= \;& c_1^2\sigma_1^2 + c_2c_1\sigma_1\sigma_2 + c_3c_1\sigma_1\sigma_3 + c_1c_2\sigma_2\sigma_1 + c_2^2\sigma_2^2 + c_3c_2\sigma_2\sigma_3 + c_1c_3\sigma_3\sigma_1 + c_2c_3\sigma_3\sigma_2 + c_3^2\sigma_3^2 \\
= \;& c_1^2(I) + c_2c_1\sigma_1\sigma_2 + c_3c_1\sigma_1\sigma_3 + c_1c_2(-\sigma_1\sigma_2) + c_2^2(I) + c_3c_2\sigma_2\sigma_3 + c_1c_3(-\sigma_1\sigma_3) + c_2c_3(-\sigma_2\sigma_3) + c_3^2(I) \\
= \;& (c_1^2 + c_2^2 + c_3^2)I
\end{aligned}
$$

⟦*Grading:* 1/2 point for free; 1/2 point for each of the four equations in part **a**; 1/2 point for each of the nine products in part **b**; 3 points for part **c**.⟧