Matrix Calisthenics

The trace

If $S = AB$ and $T = BA$, then

$$s_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj} \qquad t_{ij} = \sum_{k=1}^{N} b_{ik} a_{kj}$$

whence

$$tr(S) = \sum_{i,k=1}^{N} a_{ik} b_{ki} \qquad tr(T) = \sum_{i,k=1}^{N} b_{ik} a_{ki}$$

so

$$tr(S) = tr(T).$$

Now define $X = ABC$, so $tr(ABCD) = tr(XD) = tr(DX) = tr(DABC)$, etc.

Finally, try

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $tr(ABC) = -2$ while $tr(CBA) = 2$.

⟦*Grading:* 4 points for writing out the sums involved in $tr(AB) = tr(BA)$; 3 points for cyclic permutation argument; 3 points for finding any example where $tr(ABC) \neq tr(CBA)$.⟧

The outer product

If $P = y \otimes x$, then $p_{ij} = y_i x_j^*$, so $tr(P) = \sum_i y_i x_i^*$.

But $x \cdot y = \sum_i x_i^* y_i$, so $tr(y \otimes x) = x \cdot y$.

⟦*Grading:* 5 points for $p_{ij} = y_i x_j^*$; 5 more for $tr(y \otimes x) = x \cdot y$.⟧