# Ladder Operators for the Simple Harmonic Oscillator

**Preface.** Simple algebra shows that

$$\begin{aligned}
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^\dagger) \\
\hat{p} &= -i\sqrt{\frac{m\hbar\omega}{2}}\,(\hat{a} - \hat{a}^\dagger)
\end{aligned}$$

**a. Matrix elements.**

$$\begin{aligned}
\langle m|\hat{a}|n\rangle &= \sqrt{n}\,\delta_{m,n-1} \\
\langle m|\hat{a}^\dagger|n\rangle &= \sqrt{n+1}\,\delta_{m,n+1} \\
\langle m|\hat{x}|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right) \\
\langle m|\hat{p}|n\rangle &= -i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n}\,\delta_{m,n-1} - \sqrt{n+1}\,\delta_{m,n+1}\right)
\end{aligned}$$

⟦*Question:* What's this about all the non-vanishing matrix elements of $\hat{p}$ being imaginary? I thought that $\hat{p}$ was Hermitian, and Hermiticity is associated with real matrix elements! *Answer:* Hermiticity requires that the diagonal matrix elements be real, not the off-diagonal. And the only non-vanishing matrix elements of $\hat{p}$ are off-diagonal.⟧

The next matrix elements could be obtained by expressing the relevant operators in terms of $\hat{a}$ and $\hat{a}^\dagger$, but I prefer the following device:

$$
\begin{aligned}
& \langle m|(\hat{a} \pm \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger)|n\rangle \\
= & \sum_\ell \langle m|(\hat{a} \pm \hat{a}^\dagger)|\ell\rangle\langle \ell|(\hat{a} + \hat{a}^\dagger)|n\rangle \\
= & \sum_\ell \left(\sqrt{\ell}\,\delta_{m,\ell-1} \pm \sqrt{\ell+1}\,\delta_{m,\ell+1}\right)\left(\sqrt{n}\,\delta_{\ell,n-1} + \sqrt{n+1}\,\delta_{\ell,n+1}\right) \\
= & \sum_\ell \left[\sqrt{n}\,\delta_{\ell,n-1}\left(\sqrt{\ell}\,\delta_{m,\ell-1} \pm \sqrt{\ell+1}\,\delta_{m,\ell+1}\right) + \sqrt{n+1}\,\delta_{\ell,n+1}\left(\sqrt{\ell}\,\delta_{m,\ell-1} \pm \sqrt{\ell+1}\,\delta_{m,\ell+1}\right)\right] \\
= & \sqrt{n}\left(\sqrt{n-1}\,\delta_{m,n-2} \pm \sqrt{n}\,\delta_{m,n}\right) + \sqrt{n+1}\left(\sqrt{n+1}\,\delta_{m,n} \pm \sqrt{n+2}\,\delta_{m,n+2}\right) \\
= & \sqrt{n(n-1)}\,\delta_{m,n-2} + (n \pm n + 1)\delta_{m,n} \pm \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}
\end{aligned}
$$

Meanwhile

$$
\begin{aligned}
& \langle m|(\hat{a} \pm \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger)|n\rangle \\
= & \sum_\ell \langle m|(\hat{a} \pm \hat{a}^\dagger)|\ell\rangle\langle \ell|(\hat{a} - \hat{a}^\dagger)|n\rangle \\
= & \sum_\ell \left(\sqrt{\ell}\,\delta_{m,\ell-1} \pm \sqrt{\ell+1}\,\delta_{m,\ell+1}\right)\left(\sqrt{n}\,\delta_{\ell,n-1} - \sqrt{n+1}\,\delta_{\ell,n+1}\right) \\
= & \sum_\ell \left[\sqrt{n}\,\delta_{\ell,n-1}\left(\sqrt{\ell}\,\delta_{m,\ell-1} \pm \sqrt{\ell+1}\,\delta_{m,\ell+1}\right) - \sqrt{n+1}\,\delta_{\ell,n+1}\left(\sqrt{\ell}\,\delta_{m,\ell-1} \pm \sqrt{\ell+1}\,\delta_{m,\ell+1}\right)\right] \\
= & \sqrt{n}\left(\sqrt{n-1}\,\delta_{m,n-2} \pm \sqrt{n}\,\delta_{m,n}\right) - \sqrt{n+1}\left(\sqrt{n+1}\,\delta_{m,n} \pm \sqrt{n+2}\,\delta_{m,n+2}\right) \\
= & \sqrt{n(n-1)}\,\delta_{m,n-2} - (n \mp n + 1)\delta_{m,n} \mp \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}
\end{aligned}
$$

From these two formulas, we can read off

$$
\begin{aligned}
\langle m|\hat{x}^2|n\rangle &= \frac{\hbar}{2m\omega}\left[\sqrt{n(n-1)}\,\delta_{m,n-2} + (2n+1)\delta_{m,n} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right] \\
\langle m|\hat{p}^2|n\rangle &= -\frac{m\hbar\omega}{2}\left[\sqrt{n(n-1)}\,\delta_{m,n-2} - (2n+1)\delta_{m,n} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right] \\
\langle m|\hat{x}\hat{p}|n\rangle &= -i\frac{\hbar}{2}\left[\sqrt{n(n-1)}\,\delta_{m,n-2} - \delta_{m,n} - \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right] \\
\langle m|\hat{p}\hat{x}|n\rangle &= -i\frac{\hbar}{2}\left[\sqrt{n(n-1)}\,\delta_{m,n-2} + \delta_{m,n} - \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right]
\end{aligned}
$$

(Note: The fourth equation follows directly from the third because $[\hat{x}, \hat{p}] = i\hbar$ implies $\langle m|\hat{x}\hat{p}|n\rangle = \langle m|\hat{p}\hat{x}|n\rangle + i\hbar\delta_{m,n}$.)

Finally, because $\hat{H}|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle$,

$$\langle m|\hat{H}|n\rangle = \hbar\omega(n + \tfrac{1}{2})\delta_{m,n}.$$

**b. Kinetic and potential energy expectation values in an energy state.**

$$\langle\hat{T}\rangle = \langle n|\hat{T}|n\rangle = \frac{1}{2m}\langle n|\hat{p}^2|n\rangle = \frac{1}{2m}\frac{m\hbar\omega}{2}(2n+1) = \frac{\hbar\omega}{2}(n+\tfrac{1}{2}).$$

Because

$$\begin{aligned}
\hat{H} &= \hat{T} + \hat{U}, \\
\langle\hat{H}\rangle &= \langle\hat{T}\rangle + \langle\hat{U}\rangle, \\
\langle\hat{U}\rangle &= \langle\hat{H}\rangle - \langle\hat{T}\rangle \\
&= \hbar\omega(n+\tfrac{1}{2}) - \frac{\hbar\omega}{2}(n+\tfrac{1}{2}) \\
&= \frac{\hbar\omega}{2}(n+\tfrac{1}{2}).
\end{aligned}$$

**c. Indeterminacies for an energy state.**

In any energy state, $\langle\hat{x}\rangle = 0$ and $\langle\hat{p}\rangle = 0$, so

$$\begin{aligned}
(\Delta x)^2 &= \langle\hat{x}^2\rangle = \frac{\hbar}{2m\omega}(2n+1) \\
(\Delta p)^2 &= \langle\hat{p}^2\rangle = \frac{m\hbar\omega}{2}(2n+1).
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta x &= \sqrt{\frac{\hbar}{m\omega}}\sqrt{n+\tfrac{1}{2}} \\
\Delta p &= \sqrt{m\hbar\omega}\sqrt{n+\tfrac{1}{2}} \\
\Delta x\Delta p &= \hbar(n+\tfrac{1}{2}).
\end{aligned}$$

So the SHO ground state is a minimum indeterminacy state.

⟦*Grading:* 6 points for part **a**; 2 points for part **b**; 2 points for part **c**.⟧