Exercises on Linear Algebra

The Schwarz inequality

Following the hint, define

$$\underbrace{|\chi\rangle}_{\text{a vector}} = \underbrace{\langle\phi|\psi\rangle}_{\text{a complex number}} \underbrace{|\phi\rangle}_{\text{a vector}} - \underbrace{\langle\phi|\phi\rangle}_{\text{a real number}} \underbrace{|\psi\rangle}_{\text{a vector}} .$$

Then, using the mathematical notation for inner product,

$$\begin{aligned}
(\chi,\chi) &= \Big(\Big[(\phi,\psi)\phi - (\phi,\phi)\psi\Big], \Big[(\phi,\psi)\phi - (\phi,\phi)\psi\Big]\Big) \\
&= \Big((\phi,\psi)\phi, (\phi,\psi)\phi\Big) + \Big((\phi,\psi)\phi, -(\phi,\phi)\psi\Big) \\
&\quad + \Big(-(\phi,\phi)\psi, (\phi,\psi)\phi\Big) + \Big(-(\phi,\phi)\psi, -(\phi,\phi)\psi\Big) \\
&= (\phi,\psi)^*(\phi,\psi)\,(\phi,\phi) - (\phi,\psi)^*(\phi,\phi)\,(\phi,\psi) \\
&\quad -(\phi,\phi)^*(\phi,\psi)\,(\psi,\phi) + (\phi,\phi)^*(\phi,\phi)\,(\psi,\psi) \\
&= |(\phi,\psi)|^2\,(\phi,\phi) - |(\phi,\psi)|^2\,(\phi,\phi) \\
&\quad -|(\phi,\psi)|^2\,(\phi,\phi) + (\phi,\phi)^2\,(\psi,\psi) \\
&= -|(\phi,\psi)|^2(\phi,\phi) + (\phi,\phi)^2(\psi,\psi).
\end{aligned}$$

With this background out of the way, we notice

$$\begin{aligned}
0 &\leq (\chi,\chi) \\
0 &\leq -|(\phi,\psi)|^2(\phi,\phi) + (\phi,\phi)^2(\psi,\psi) \\
0 &\leq -|(\phi,\psi)|^2 + (\phi,\phi)(\psi,\psi) \\
|(\phi,\psi)|^2 &\leq (\phi,\phi)(\psi,\psi) \\
|(\phi,\psi)| &\leq \sqrt{(\phi,\phi)(\psi,\psi)}.
\end{aligned}$$

Eigenproblem for functions of operators, I

The inductive chain:

$n=0$:	$c\hat{A}^0 = c\hat{1}$,	and any vector is an eigenvector of this operator with eigenvalue c.
$n=1$:	$c\hat{A}^1 = c\hat{A}$,	and by hypothesis this operator has eigenvectors $\{|a_i\rangle\}$ with eigenvalues $\{ca_i\}$.

Now assume that the operator $c\hat{A}^{n-1}$ has eigenvectors $\{|a_i\rangle\}$ with eigenvalues $\{ca_i^{n-1}\}$. Then

$$\begin{aligned} c\hat{A}^n|a_i\rangle &= (c\hat{A}^{n-1})(\hat{A}|a_i\rangle) \qquad [\![\text{by hypothesis}\ldots]\!] \\ &= (c\hat{A}^{n-1})(a_i|a_i\rangle) \\ &= a_i(c\hat{A}^{n-1}|a_i\rangle) \qquad [\![\text{by inductive assumption}\ldots]\!] \\ &= a_i(ca_i^{n-1}|a_i\rangle) \\ &= ca_i^n|a_i\rangle \end{aligned}$$

and we're done.

Eigenproblem for functions of operators, II

$$\begin{aligned} f(\hat{A})|a_i\rangle &= \sum_{n=0}^{\infty} c_n \hat{A}^n |a_i\rangle \\ &= \sum_{n=0}^{\infty} c_n a_i^n |a_i\rangle \\ &= \left[\sum_{n=0}^{\infty} c_n a_i^n\right] |a_i\rangle \\ &= f(a_i)|a_i\rangle \end{aligned}$$