Quantum Mechanics 2023

Model Solutions for Final Exam

1. Tent wavefunction

From symmetry, $\langle \hat{x} \rangle = 0$. To ensure normalization,

$$\begin{aligned}
1 &= \int_{-\infty}^{+\infty} |\psi(x)|^2 \, dx \\
&= 2 \int_0^{L/2} a^2 \left(x - \frac{L}{2} \right)^2 dx \\
&= 2a^2 \left(\frac{L}{2} \right)^3 \int_0^1 (u-1)^2 \, du \\
&= a^2 \frac{L^3}{4} \int_0^1 (u^2 - 2u + 1) \, du \\
&= a^2 \frac{L^3}{4} \left[\frac{1}{3} u^3 - 2 \frac{1}{2} u^2 + u \right]_0^1 \\
&= a^2 \frac{L^3}{4} \frac{1}{3} \\
a^2 &= \frac{12}{L^3}.
\end{aligned}$$

The indeterminacy is given through

$$\begin{aligned}
(\Delta x)^2 &= \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 \\
&= \int_{-\infty}^{+\infty} x^2 |\psi(x)|^2 \, dx \\
&= 2 \int_0^{L/2} x^2 a^2 \left(x - \frac{L}{2} \right)^2 dx \\
&= 2a^2 \left(\frac{L}{2} \right)^5 \int_0^1 u^2 (u-1)^2 \, du \\
&= a^2 \frac{L^5}{2^4} \int_0^1 (u^4 - 2u^3 + u^2) \, du \\
&= a^2 \frac{L^5}{2^4} \left[\frac{1}{5} u^5 - 2 \frac{1}{4} u^4 + \frac{1}{3} u^3 \right]_0^1 \\
&= a^2 \frac{L^5}{2^4} \frac{1}{30} \\
&= \frac{12}{L^3} \frac{L^5}{2^4} \frac{1}{30} = \frac{1}{40} L^2.
\end{aligned}$$

So

$$\Delta x = \frac{1}{2\sqrt{10}} L.$$

2. Muonic hydrogen

Any distance in any Coulomb problem is proportional to the characteristic length $\hbar^2/kM$ (see notes, equation (14.56)). The muonic hydrogen Coulomb problem is exactly the same as the electronic hydrogen Coulomb problem except that M increases by a factor of 206.8. Thus all lengths decrease by a factor of 206.8, and the mean distance in question is

$$\frac{0.952 \text{ nm}}{206.8} = 0.00460 \text{ nm}.$$

3 and 4 combined, time evolution

The state $\frac{1}{\sqrt{2}}[|3\rangle + i|4\rangle]$ evolves in time to

$$|\psi(t)\rangle = \tfrac{1}{\sqrt{2}}[e^{-(i/\hbar)E_3 t}|3\rangle + ie^{-(i/\hbar)E_4 t}|4\rangle],$$

with bra

$$\langle\psi(t)| = \tfrac{1}{\sqrt{2}}[e^{+(i/\hbar)E_3 t}\langle 3| - ie^{+(i/\hbar)E_4 t}|4\rangle].$$

So the mean value of position is

$$\begin{aligned}
\langle x\rangle_t &= \langle\psi(t)|\hat{x}|\psi(t)\rangle \\
&= \tfrac{1}{2}\Big[e^{+(i/\hbar)E_3 t}e^{-(i/\hbar)E_3 t}\langle 3|\hat{x}|3\rangle + e^{+(i/\hbar)E_3 t}ie^{-(i/\hbar)E_4 t}\langle 3|\hat{x}|4\rangle \\
&\qquad - ie^{+(i/\hbar)E_4 t}e^{-(i/\hbar)E_3 t}\langle 4|\hat{x}|3\rangle - ie^{+(i/\hbar)E_4 t}ie^{-(i/\hbar)E_4 t}\langle 4|\hat{x}|4\rangle\Big] \\
&= \tfrac{1}{2}\Big[\langle 3|\hat{x}|3\rangle + ie^{-(i/\hbar)(E_4-E_3)t}\langle 3|\hat{x}|4\rangle \\
&\qquad - ie^{+(i/\hbar)(E_4-E_3)t}\langle 4|\hat{x}|3\rangle + \langle 4|\hat{x}|4\rangle\Big] \\
&= \tfrac{1}{2}\Big[\langle 3|\hat{x}|3\rangle + \langle 4|\hat{x}|4\rangle + 2\,\Re e\Big\{ie^{-(i/\hbar)(E_4-E_3)t}\langle 3|\hat{x}|4\rangle\Big\}\Big] \\
&= \tfrac{1}{2}\left[\langle 3|\hat{x}|3\rangle + \langle 4|\hat{x}|4\rangle\right] + \sin((E_4-E_3)t/\hbar)\,\langle 3|\hat{x}|4\rangle
\end{aligned}$$

where in the last step we've used the fact that $\langle 3|\hat{x}|4\rangle$ is pure real.

3. Time evolution in the SHO

For the SHO, the position origin is conventionally taken at the bottom of the well. The mean values $\langle 3|\hat{x}|3\rangle$ and $\langle 4|\hat{x}|4\rangle$ are both zero. The energy difference is $E_4 - E_3 = \hbar\omega$. The matrix element is

$$\begin{aligned}
\langle 3|\hat{x}|4\rangle &= \sqrt{\hbar/2m\omega}\,\langle 3|(\hat{a}+\hat{a}^\dagger)|4\rangle \\
&= \sqrt{\hbar/2m\omega}\,\langle 3|\hat{a}|4\rangle \\
&= \sqrt{\hbar/2m\omega}\,\sqrt{4}\langle 3|3\rangle \\
&= 2\sqrt{\hbar/2m\omega}.
\end{aligned}$$

Putting this all together,

$$\langle\hat{x}\rangle_t = \sqrt{\frac{2\hbar}{m\omega}}\sin(\omega t).$$

4. Time evolution in the ISW

For the ISW, the position origin is conventionally taken at the left wall of the well. The mean values $\langle 3|\hat{x}|3\rangle$ and $\langle 4|\hat{x}|4\rangle$ are both $L/2$. The energy difference is

$$E_4 - E_3 = 4^2 \frac{\pi^2\hbar^2}{2mL^2} - 3^2 \frac{\pi^2\hbar^2}{2mL^2} = 7\frac{\pi^2\hbar^2}{2mL^2}.$$

The matrix element is

$$\begin{aligned}
\langle 3|\hat{x}|4\rangle &= \frac{2}{L}\int_0^L \sin(3\pi x/L)x\sin(4\pi x/L)\,dx \\
&= \frac{2}{L}\left(\frac{L}{\pi}\right)^2 \int_0^\pi \theta\sin(3\theta)\sin(4\theta)\,d\theta \\
&= \frac{2}{L}\left(\frac{L}{\pi}\right)^2 \left(-\frac{48}{49}\right) = -\frac{96}{49\pi^2}L.
\end{aligned}$$

Putting this all together,

$$\langle \hat{x}\rangle_t = \frac{1}{2}L - \frac{96}{49\pi^2}L\sin\left(\frac{7\pi^2\hbar}{2mL^2}t\right).$$