Quantum Mechanics

Model Solutions for Sample Exam for Final Examination

1. (a) Expectation value. In terms of ladder operators,

$$\hat{J}_x = \tfrac{1}{2}(\hat{J}_+ + \hat{J}_-).$$

Thus

$$\langle \hat{J}_x \rangle = \tfrac{1}{2}(\langle j, m|\hat{J}_+|j, m\rangle + \langle j, m|\hat{J}_-|j, m\rangle) = 0.$$

(b) Uncertainty. We have

$$(\Delta J_x)^2 = \langle \hat{J}_x^2 \rangle - \langle \hat{J}_x \rangle^2 = \langle \hat{J}_x^2 \rangle.$$

But

$$\langle \hat{J}^2 \rangle = \langle \hat{J}_x^2 \rangle + \langle \hat{J}_y^2 \rangle + \langle \hat{J}_z^2 \rangle.$$

By symmetry, $\langle \hat{J}_x^2 \rangle = \langle \hat{J}_y^2 \rangle$, so

$$(\Delta J_x)^2 = \langle \hat{J}_x^2 \rangle = \tfrac{1}{2}[\langle \hat{J}^2 \rangle - \langle \hat{J}_z^2 \rangle] = \tfrac{1}{2}[\hbar^2 j(j+1) - (\hbar m)^2]$$

whence

$$\Delta J_x = \hbar\sqrt{\tfrac{1}{2}[j(j+1) - m^2]}.$$

2. Scaling. We've seen that, within the Coulomb problem with mass M and interaction $V(r) = -k/r$, there is only one quantity with the dimensions of length, namely $\hbar^2/kM$. (See *The Physics of QM* equation 14.72.) Any quantity with the dimensions of length (such as the mean separation in the energy eigenstate with $n = 4$, $\ell = 3$, and $m = 0$) must take the form

$$(\text{dimensionless number})\frac{\hbar^2}{kM}.$$

For the hydrogen atom problem, $M = m_e$ (ignoring nuclear motion) and $k = e^2/4\pi\epsilon_0$. For the helium ion problem, $M = m_e$ (ignoring nuclear motion) and $k = 2e^2/4\pi\epsilon_0$. Hence any length calculated for the helium ion problem is half the corresponding length for the hydrogen atom problem. The length in question is thus $\frac{1}{2}(0.952 \text{ nm}) = 0.476$ nm.

⟦It so happens that for the length in question, the dimensionless number is 18. But it's a long calculation to find that 18, and you don't need to do it in order to solve the problem.⟧

3. Spin-$\frac{1}{2}$ system.

a. We have

$$|\psi(0)\rangle = \psi_+|\uparrow\rangle + \psi_-|\downarrow\rangle$$

where $|\psi_+|^2 = \frac{1}{2}$, $|\psi_-|^2 = \frac{1}{2}$, so $\psi_+ = \frac{1}{\sqrt{2}}e^{i\delta_+}$, $\psi_- = \frac{1}{\sqrt{2}}e^{i\delta_-}$. Use overall phase freedom to select $\delta_+ = 0$ and we have

$$|\psi(0)\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\rangle + e^{i\delta_-}|\downarrow\rangle).$$

b. The Hamiltonian is diagonal, so

$$|\psi(t)\rangle = \tfrac{1}{\sqrt{2}}(e^{-iat/\hbar}|\uparrow\rangle + e^{-ibt/\hbar}e^{i\delta_-}|\downarrow\rangle).$$

4. The virial theorem.

a. In an energy eigenstate (stationary state), any observable is time constant, so

$$\frac{d}{dt}\langle\hat{x}\hat{p}\rangle = 0.$$

b. As always

$$\frac{d}{dt}\langle\hat{x}\hat{p}\rangle = -\frac{i}{\hbar}\langle[\hat{x}\hat{p},\hat{H}]\rangle.$$

c.

$$\begin{aligned}
[\hat{x}\hat{p},\hat{H}] &= [\hat{x}\hat{p},\hat{p}^2/2m + V(\hat{x})] \\
&= [\hat{x}\hat{p},\hat{p}^2/2m] + [\hat{x}\hat{p},V(\hat{x})] \\
&= \hat{x}[\hat{p},\hat{p}^2/2m] + [\hat{x},\hat{p}^2/2m]\hat{p} + \hat{x}[\hat{p},V(\hat{x})] + [\hat{x},V(\hat{x})]\hat{p} \\
&= 0 + [\hat{x},\hat{p}^2]\hat{p}/2m + \hat{x}[\hat{p},V(\hat{x})] + 0 \\
&= (\hat{p}[\hat{x},\hat{p}] + [\hat{x},\hat{p}]\hat{p})\,\hat{p}/2m + \hat{x}[\hat{p},V(\hat{x})] \\
&= (2i\hbar\hat{p})\,\hat{p}/2m + \hat{x}[\hat{p},V(\hat{x})] \\
&= 2i\hbar\widehat{\mathrm{KE}} + \hat{x}[\hat{p},V(\hat{x})].
\end{aligned}$$

Find $[\hat{p},V(\hat{x})]$ by evaluating the commutator in the position representation:

$$\begin{aligned}
[\hat{p},V(\hat{x})]|\psi\rangle &= \hat{p}V(\hat{x})|\psi\rangle - V(\hat{x})\hat{p}|\psi\rangle \\
&\doteq -i\hbar\left[\frac{\partial}{\partial x}V(x)\psi(x) - V(x)\frac{\partial}{\partial x}\psi(x)\right] \\
&= -i\hbar\left[\frac{\partial V(x)}{\partial x}\psi(x) + V(x)\frac{\partial\psi(x)}{\partial x} - V(x)\frac{\partial\psi(x)}{\partial x}\right] \\
&= i\hbar F(x)\psi(x)
\end{aligned}$$

where $F(x) = -\dfrac{\partial V(x)}{\partial x}$. Because this holds for arbitrary $|\psi\rangle$, $[\hat{p},V(\hat{x})] = i\hbar F(\hat{x})$. Thus, going back to our first chain of deductions,

$$[\hat{x}\hat{p},\hat{H}] = 2i\hbar\widehat{\mathrm{KE}} + i\hbar\hat{x}F(\hat{x}).$$

Add this to the results of parts (a) and (b) to find that, for any energy eigenstate

$$\frac{d}{dt}\langle\hat{x}\hat{p}\rangle = -\frac{i}{\hbar}\langle[\hat{x}\hat{p},\hat{H}]\rangle = \langle 2\widehat{\mathrm{KE}} + \hat{x}F(\hat{x})\rangle = 0,$$

whence

$$2\langle\widehat{\mathrm{KE}}\rangle = -\langle\hat{x}F(\hat{x})\rangle$$

. . . the virial theorem!