Quantum Mechanics

Model Solutions for Sample Exam for Second Examination

1. According to equation (5.35) in the text,

$$|\psi(t)\rangle = e^{-(i/\hbar)Et}\left[e^{-(i/\hbar)(-A)t}\langle e_1|\psi(0)\rangle|e_1\rangle + e^{-(i/\hbar)(+A)t}\langle e_2|\psi(0)\rangle|e_2\rangle\right].$$

where $|\psi(0)\rangle = |d\rangle$. But from (5.34), $\langle e_1|d\rangle = -\frac{1}{\sqrt{2}}e^{+i\phi}$ and $\langle e_2|d\rangle = +\frac{1}{\sqrt{2}}e^{+i\phi}$ so

$$|\psi(t)\rangle = e^{-(i/\hbar)Et}\left(\tfrac{1}{\sqrt{2}}e^{+i\phi}\right)\left[-e^{-(i/\hbar)(-A)t}|e_1\rangle + e^{-(i/\hbar)(+A)t}|e_2\rangle\right].$$

We must first find the amplitude

$$\begin{aligned}
\langle d|\psi(t)\rangle &= e^{-(i/\hbar)Et}\left(\tfrac{1}{\sqrt{2}}e^{+i\phi}\right)\left[-e^{-(i/\hbar)(-A)t}\langle d|e_1\rangle + e^{-(i/\hbar)(+A)t}\langle d|e_2\rangle\right] \\
&= e^{-(i/\hbar)Et}\left(\tfrac{1}{\sqrt{2}}e^{+i\phi}\right)\left(\tfrac{1}{\sqrt{2}}e^{-i\phi}\right)\left[-e^{-(i/\hbar)(-A)t}(-1) + e^{-(i/\hbar)(+A)t}(+1)\right] \\
&= e^{-(i/\hbar)Et}\left(\frac{1}{2}\right)\left[e^{-(i/\hbar)(-A)t} + e^{-(i/\hbar)(+A)t}\right] \\
&= e^{-(i/\hbar)Et}\cos\left(\frac{A}{\hbar}t\right).
\end{aligned}$$

Thus the probability of finding the nitrogen atom “down” is

$$\cos^2\left(\frac{A}{\hbar}t\right).$$

2. From the commutator,

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \quad \text{so} \quad \langle\hat{x}\hat{p}\rangle - \langle\hat{p}\hat{x}\rangle = i\hbar.$$

But

$$(\hat{x}\hat{p})^\dagger = \hat{p}^\dagger\hat{x}^\dagger = \hat{p}\hat{x} \quad \text{so} \quad \langle\hat{x}\hat{p}\rangle^* = \langle\hat{p}\hat{x}\rangle.$$

Together

$$\langle\hat{x}\hat{p}\rangle - \langle\hat{x}\hat{p}\rangle^* = i\hbar \quad \text{so} \quad 2i\Im m\{\langle\hat{x}\hat{p}\rangle\} = i\hbar \quad \text{so} \quad \Im m\{\langle\hat{x}\hat{p}\rangle\} = \tfrac{1}{2}\hbar.$$

3. We have

$$\hat{A}\hat{B} - \hat{B}\hat{A} = c\hat{B} \quad \text{so} \quad \hat{A}\hat{B}|a\rangle - a\hat{B}|a\rangle = c\hat{B}|a\rangle \quad \text{so} \quad \hat{A}(\hat{B}|a\rangle) = (a+c)(\hat{B}|a\rangle).$$

That last equation says that $\hat{B}|a\rangle$ is an eigenvector of $\hat{A}$ with eigenvalue $a+c$. Hence it is also an eigenvector of $\hat{A}^3$ with eigenvalue $(a+c)^3$.

4. Lorentzian wavepacket

For

$$\psi(x) = \frac{A}{x^2+\gamma^2} e^{ikx},$$

the normalization is given through

$$\begin{aligned}
1 &= \int_{-\infty}^{+\infty} \psi^*(x)\psi(x)\,dx \\
&= A^2 \int_{-\infty}^{+\infty} \frac{1}{(x^2+\gamma^2)^2}\,dx \qquad [\![\text{Use Dwight 120.2 giving}\ldots]\!] \\
&= A^2 \left[\frac{x}{2\gamma^2(x^2+\gamma^2)} + \frac{1}{2\gamma^3}\arctan\frac{x}{\gamma} \right]_{-\infty}^{+\infty} \\
&= A^2 \left[\frac{1}{2\gamma^3}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)\right] = A^2 \frac{\pi}{2\gamma^3}
\end{aligned}$$

so $A^2 = 2\gamma^3/\pi$.

The mean kinetic energy is given through

$$\langle \text{KE} \rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \psi^*(x) \frac{d^2}{d^2x}\psi(x)\,dx = +\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \frac{d\psi^*(x)}{dx}\frac{d\psi(x)}{dx}\,dx$$

where we have used integration by parts to cast the integral into a more symmetric form. (And to avoid taking a second derivative!) Now

$$\begin{aligned}
\frac{d\psi(x)}{dx} &= -\frac{A\,2x}{(x^2+\gamma^2)^2}e^{ikx} + ik\psi(x) \\
\frac{d\psi^*(x)}{dx} &= -\frac{A\,2x}{(x^2+\gamma^2)^2}e^{-ikx} - ik\psi^*(x)
\end{aligned}$$

so

$$\begin{aligned}
\langle \text{KE} \rangle &= \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty} \left[\frac{A\,2x}{(x^2+\gamma^2)^2}e^{-ikx} + ik\psi^*(x)\right]\left[\frac{A\,2x}{(x^2+\gamma^2)^2}e^{ikx} - ik\psi(x)\right] dx \\
&= \frac{\hbar^2}{2m}\left[\int_{-\infty}^{+\infty} \left(\frac{A\,2x}{(x^2+\gamma^2)^2}\right)^2 dx + \int_{-\infty}^{+\infty} \left(\frac{A\,2x}{(x^2+\gamma^2)^2}e^{-ikx}\right)\left(-ik\frac{A}{x^2+\gamma^2}e^{ikx}\right) dx \right. \\
&\quad \left. + \int_{-\infty}^{+\infty} \left(+ik\frac{A}{x^2+\gamma^2}e^{-ikx}\right)\left(\frac{A\,2x}{(x^2+\gamma^2)^2}e^{ikx}\right) dx + k^2 \int_{-\infty}^{+\infty} \psi^*(x)\psi(x)\,dx \right]
\end{aligned}$$

Of these four integrals, the last one is just the normalization integral, so it is one. The second and third integrals have odd integrands, so they are zero. We're left with

$$\langle \text{KE} \rangle = \frac{\hbar^2}{2m}\left[4A^2 \int_{-\infty}^{+\infty} \frac{x^2}{(x^2+\gamma^2)^4}dx + k^2 \right].$$

Now, using Dwight 122.4,

$$\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+\gamma^2)^4}dx = \left[-\frac{x}{6(x^2+\gamma^2)^3} + \frac{x}{24\gamma^2(x^2+\gamma^2)^2} + \frac{x}{16\gamma^4(x^2+\gamma^2)} + \frac{1}{16\gamma^5}\arctan\frac{x}{\gamma}\right]_{-\infty}^{+\infty} = \frac{\pi}{16\gamma^5},$$

whence

$$\langle \text{KE} \rangle = \frac{\hbar^2}{2m}\left[k^2 + \frac{1}{2\gamma^2}\right].$$