Quantum Mechanics

Model Solutions for Sample Exam for First Examination

1. State representations

The representation of $|z-\rangle$ in the basis $\{|\theta+\rangle, |\theta-\rangle\}$ is

$$\begin{pmatrix} \langle\theta+|z-\rangle \\ \langle\theta-|z-\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta/2 \\ \cos\theta/2 \end{pmatrix}.$$

2. Photon polarization

Using the amplitudes determined in the third problem on photon polarization,

$$a(\theta) = \langle x|\theta\rangle = \cos\theta \qquad b(\theta) = \langle y|\theta\rangle = \sin\theta.$$

3. Diagonalizing the Pauli matrices

1. Find eigenvalues:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \det(\sigma_1 - \lambda I) = 0; \quad \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0; \quad \lambda^2 - 1 = 0; \quad \lambda = \pm 1.$$

Find eigenvectors for $\lambda_1 = +1$:

$$(\sigma_1 - \lambda_1 I)\begin{pmatrix} x \\ y \end{pmatrix} = 0; \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad y = x; \quad e_1 = \begin{pmatrix} x \\ x \end{pmatrix}; \quad \text{normalized } e_1 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find eigenvectors for $\lambda_2 = -1$:

$$(\sigma_1 - \lambda_2 I)\begin{pmatrix} x \\ y \end{pmatrix} = 0; \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad y = -x; \quad e_2 = \begin{pmatrix} x \\ -x \end{pmatrix}; \quad \text{normalized } e_2 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

2. Find eigenvalues:

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \det(\sigma_2 - \lambda I) = 0; \quad \det\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = 0; \quad \lambda^2 - 1 = 0; \quad \lambda = \pm 1.$$

Find eigenvectors for $\lambda_1 = +1$:

$$(\sigma_2 - \lambda_1 I)\begin{pmatrix} x \\ y \end{pmatrix} = 0; \quad \begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad y = ix; \quad e_1 = \begin{pmatrix} x \\ ix \end{pmatrix}; \quad \text{normalized } e_1 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Find eigenvectors for $\lambda_2 = -1$:

$$(\sigma_2 - \lambda_2 I)\begin{pmatrix} x \\ y \end{pmatrix} = 0; \quad \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad y = -ix; \quad e_2 = \begin{pmatrix} x \\ -ix \end{pmatrix}; \quad \text{normalized } e_2 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}.$$

3.

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Already diagonal:

$$\lambda_1 = 1; \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \lambda_2 = -1; \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

4. Math!

Since $(a, \hat{A}a) = (a, \hat{A}a)^* = (\hat{A}a, a)$ for all a, it holds for $a = b + c$:

$$\begin{aligned}
(b + c, \hat{A}b + \hat{A}c) &= (\hat{A}b + \hat{A}c, b + c) && (1)\\
(b, \hat{A}b) + (b, \hat{A}c) + (c, \hat{A}b) + (c, \hat{A}c) &= (\hat{A}b, b) + (\hat{A}b, c) + (\hat{A}c, b) + (\hat{A}c, c) && (2)\\
(b, \hat{A}c) + (c, \hat{A}b) &= (\hat{A}b, c) + (\hat{A}c, b) && (3)
\end{aligned}$$

[Where the step from (2) to (3) uses the fact that $(b, \hat{A}b) = (\hat{A}b, b)$ for all b.]

Meanwhile, it also holds for $a = b + ic$:

$$\begin{aligned}
(b + ic, \hat{A}b + i\hat{A}c) &= (\hat{A}b + i\hat{A}c, b + ic) && (4)\\
(b, \hat{A}b) + i(b, \hat{A}c) - i(c, \hat{A}b) + (c, \hat{A}c) &= (\hat{A}b, b) + i(\hat{A}b, c) - i(\hat{A}c, b) + (\hat{A}c, c) && (5)\\
(b, \hat{A}c) - (c, \hat{A}b) &= (\hat{A}b, c) - (\hat{A}c, b) && (6)
\end{aligned}$$

[Where the step from (5) to (6) uses the fact that $(b, \hat{A}b) = (\hat{A}b, b)$ for all b.]

Now add equations (3) and (6), then divide by two:

$$\begin{aligned}
(b, \hat{A}c) &= (\hat{A}b, c) && (7)\\
(b, \hat{A}c) &= (c, \hat{A}b)^* && (8)
\end{aligned}$$

and this holds for all b and c, so $\hat{A}$ is Hermitian.