Degenerate Perturbation Theory in a Two-State System

a. Using the initial basis (call it $\{|1_{\text{init}}\rangle, |2_{\text{init}}\rangle\}$):

$$\begin{aligned}
\langle 1_{\text{init}}|\hat{H}'|1_{\text{init}}\rangle &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} a_3 & a_1 \\ a_1 & -a_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_3 \\
\langle 2_{\text{init}}|\hat{H}'|2_{\text{init}}\rangle &= \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} a_3 & a_1 \\ a_1 & -a_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -a_3
\end{aligned}$$

Using the second basis (call it $\{|1_{\text{second}}\rangle, |2_{\text{second}}\rangle\}$):

$$\begin{aligned}
\langle 1_{\text{second}}|\hat{H}'|1_{\text{second}}\rangle &= \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} a_3 & a_1 \\ a_1 & -a_3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = a_1 \\
\langle 2_{\text{second}}|\hat{H}'|2_{\text{second}}\rangle &= \frac{1}{2}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} a_3 & a_1 \\ a_1 & -a_3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -a_1
\end{aligned}$$

b. The eigenvalues of $\hat{H}'$ are $\pm\sqrt{a_1^2 + a_3^2}$, so the representation of $\hat{H}'$ in the diagonalizing basis is

$$\begin{pmatrix} \sqrt{a_1^2 + a_3^2} & 0 \\ 0 & -\sqrt{a_1^2 + a_3^2} \end{pmatrix}.$$

c. The representation of $\hat{H}^{(0)}$ in *any* basis is

$$\begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}.$$

d. The representation of the full Hamiltonian $\hat{H}^{(0)} + \hat{H}'$ in the diagonalizing basis is the sum of the above two matrices, namely

$$\begin{pmatrix} a_0 + \sqrt{a_1^2 + a_3^2} & 0 \\ 0 & a_0 - \sqrt{a_1^2 + a_3^2} \end{pmatrix}.$$

The full Hamiltonian is diagonal in this basis. Its exact eigenvalues are

$$E_n = a_0 \pm \sqrt{a_1^2 + a_3^2}.$$

e. The full matrix is diagonal in this basis, so the basis states are exact energy eigenstates.

f. Tedious but straightforward.

⟦*Grading:* 2 points for part **a**, 2 points for **b**, 1 point for **c**, 2 points for **d**, 1 point for **e**, 2 points for **f**.⟧