Continuous Systems

Convention: In these solutions, all the integrals run from $-\infty$ to $+\infty$.

The states $\{|p\rangle\}$ constitute a continuous basis

a. If

$$\hat{A} = \int_{-\infty}^{\infty} |p\rangle\langle p|\, dp$$

then, for arbitrary states $|\phi\rangle$ and $|\psi\rangle$,

$$\begin{aligned}
\langle\phi|\hat{A}|\psi\rangle &= \int \langle\phi|p\rangle\langle p|\psi\rangle\, dp \\
&= \int dp\, \langle\phi|\hat{1}|p\rangle\langle p|\hat{1}|\psi\rangle \\
&= \int dx \int dp \int dx'\, \langle\phi|x\rangle\langle x|p\rangle\langle p|x'\rangle\langle x'|\psi\rangle \\
&= \int dx \int dp \int dx'\, \phi^*(x) C e^{i(p/\hbar)x}\, C^* e^{-i(p/\hbar)x'}\, \psi(x') \\
&= CC^* \int dx \int dp \int dx'\, \phi^*(x) e^{i(p/\hbar)(x-x')}\, \psi(x') \\
&= |C|^2 \int dx \int dx'\, \phi^*(x)\psi(x') \left[\int dp\, e^{i(p/\hbar)(x-x')}\right] \\
&= |C|^2 \int dx \int dx'\, \phi^*(x)\psi(x') \left[\hbar \int dk\, e^{ik(x-x')}\right] \\
&= |C|^2 \int dx \int dx'\, \phi^*(x)\psi(x') \left[\hbar\, 2\pi\delta(x-x')\right] \\
&= 2\pi\hbar|C|^2 \int dx \int dx'\, \phi^*(x)\psi(x')\delta(x-x') \\
&= 2\pi\hbar|C|^2 \int dx\, \phi^*(x)\psi(x) \\
&= 2\pi\hbar|C|^2 \langle\phi|\psi\rangle.
\end{aligned}$$

Because $|\phi\rangle$ and $|\psi\rangle$ are arbitrary, we conclude that

$$\hat{A} = 2\pi\hbar|C|^2\hat{1}.$$

The choice $C = 1/\sqrt{2\pi\hbar}$ makes

$$\hat{A} = \int |p\rangle\langle p|\, dp = \hat{1}.$$

b.

$$
\begin{aligned}
\langle p|p'\rangle &= \int dx\, \langle p|x\rangle\langle x|p'\rangle \\
&= \int dx\, C^* e^{-i(p/\hbar)x}\, C e^{i(p'/\hbar)x} \\
&= |C|^2 \int dx\, e^{i(x/\hbar)(p'-p)} \qquad [\![\ldots \text{use } u = x/\hbar\ \ldots]\!] \\
&= |C|^2\hbar \int du\, e^{iu(p'-p)} \\
&= 2\pi\hbar|C|^2\delta(p-p').
\end{aligned}
$$

The choice $C = 1/\sqrt{2\pi\hbar}$ is again convenient leading to

$$\langle p|p'\rangle = \delta(p-p').$$

[[*Grading:* 5 points for part a, 5 points for part b.]]

Peculiarities of continuous basis states

Set

$$\chi(x) = \langle x|x'\rangle = \delta(x-x')$$

so that

$$\int \chi^*(x)\chi(x)\,dx = \int \delta(x-x')\delta(x-x')\,dx = \delta(0) = \infty.$$

Set

$$\pi(x) = \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i(p/\hbar)x}$$

so that

$$\int \pi^*(x)\pi(x)\,dx = \frac{1}{2\pi\hbar}\int 1\,dx = \infty.$$

[[*Grading:* 5 points for each ∞.]]

Hermiticity of the momentum operator

$$\begin{aligned}
\langle\phi|\hat{p}|\psi\rangle &= \int \langle\phi|x\rangle\langle x|\hat{p}|\psi\rangle\, dx \\
&= \int \phi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x)\, dx \\
&= -i\hbar\left[\int_{-\infty}^{+\infty}\phi^*(x)\frac{d\psi(x)}{dx}\, dx\right] \qquad [\![\ldots\text{integrate by parts}\ldots]\!] \\
&= -i\hbar\left[\phi^*(x)\psi(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\psi(x)\frac{d\phi^*(x)}{dx}\, dx\right] \qquad [\![\ldots\text{left piece is zero by assumption}\ldots]\!] \\
&= \left[i\hbar\int\psi(x)\frac{d\phi^*(x)}{dx}\, dx\right] \\
&= \left[-i\hbar\int\psi^*(x)\frac{d\phi(x)}{dx}\, dx\right]^* \\
&= \langle\psi|\hat{p}|\phi\rangle^*
\end{aligned}$$

Thus $\hat{p}$ is Hermitian as long is it operates on states whose wavefunctions vanish at $\pm\infty$.

⟦*Grading:* 5 points for realizing you need to use integration by parts, 5 points for using it correctly.⟧

Commutator of $\hat{x}$ and $\hat{p}$

Define $|\phi_1\rangle = \hat{x}|\psi\rangle$ so that

$$\begin{aligned}
\langle x|\hat{p}\hat{x}|\psi\rangle &= \langle x|\hat{p}|\phi_1\rangle \\
&= -i\hbar\frac{\partial}{\partial x}\langle x|\phi_1\rangle \\
&= -i\hbar\frac{\partial}{\partial x}(\langle x|\hat{x}|\psi\rangle) \\
&= -i\hbar\frac{\partial}{\partial x}(x\psi(x)) \\
&= -i\hbar\psi(x) - i\hbar x\frac{\partial\psi(x)}{\partial x}.
\end{aligned}$$

Meanwhile, define $|\phi_2\rangle = \hat{p}|\psi\rangle$ so that

$$\begin{aligned}
\langle x|\hat{x}\hat{p}|\psi\rangle &= \langle x|\hat{x}|\phi_2\rangle \\
&= x\langle x|\phi_2\rangle \\
&= x\langle x|\hat{p}|\psi\rangle \\
&= x\left(-i\hbar\frac{\partial\psi(x)}{\partial x}\right).
\end{aligned}$$

Hence

$$\langle x|[\hat{x},\hat{p}]|\psi\rangle = -i\hbar x\frac{\partial\psi(x)}{\partial x} + i\hbar\psi(x) + i\hbar x\frac{\partial\psi(x)}{\partial x} = i\hbar\langle x|\psi\rangle.$$

Now consider the commutator operator between two arbitary states

$$\begin{aligned}
\langle\chi|[\hat{x},\hat{p}]|\psi\rangle &= \int dx\,\langle\chi|x\rangle\langle x|[\hat{x},\hat{p}]|\psi\rangle \\
&= i\hbar\int dx\,\langle\chi|x\rangle\langle x|\psi\rangle \\
&= i\hbar\langle\chi|\psi\rangle.
\end{aligned}$$

Because $|\chi\rangle$ and $|\psi\rangle$ are arbitary,

$$[\hat{x},\hat{p}] = i\hbar\hat{1}.$$

The usual convention is that the identity operator $\hat{1}$ is understood, so this is written as

$$[\hat{x},\hat{p}] = i\hbar.$$

⟦*Grading:* There are many ways to do this problem. All correct ways, even if inelegant, earn 10 points.⟧

Momentum representation of the Schrödinger equation

a.

$$\begin{aligned}
\langle p|\hat{H}|\psi(t)\rangle &= \langle\psi(t)|\hat{H}|p\rangle^* \\
&= \langle\psi(t)|(\hat{p}^2/2m)|p\rangle^* + \langle\psi(t)|\hat{V}|p\rangle^* \\
&= (p^2/2m)\langle\psi(t)|p\rangle^* + \langle\psi(t)|\hat{V}|p\rangle^* \\
&= (p^2/2m)\langle p|\psi(t)\rangle + \langle p|\hat{V}|\psi(t)\rangle \\
&= \frac{p^2}{2m}\tilde{\psi}(p;t) + \langle p|\hat{V}|\psi(t)\rangle.
\end{aligned}$$

b.

$$\begin{aligned}
\langle p|\hat{V}|\psi(t)\rangle &= \langle p|\hat{1}\hat{V}|\psi(t)\rangle \\
&= \int dx\,\langle p|x\rangle\langle x|\hat{V}|\psi(t)\rangle \\
&= \int dx\,\left(\frac{e^{-i(p/\hbar)x}}{\sqrt{2\pi\hbar}}\right)(V(x)\psi(x;t)) \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int dx\,e^{-i(p/\hbar)x}\,V(x)\psi(x;t).
\end{aligned}$$

c. The function $V(x)$ has the dimensions [energy], but the function $\tilde{V}(p)$ has the dimensions

$$[\text{energy}]\sqrt{\frac{[\text{length}]}{[\text{momentum}]}} = [\text{energy}]\sqrt{\frac{[\text{time}]}{[\text{mass}]}}.$$

Proof in the"Fourier transform style":

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi\hbar}}\int dp\,e^{i(p/\hbar)x}\,\tilde{V}(p) \qquad [\![\text{Use definition (6.17a) …}]\!] \\
=\;& \frac{1}{\sqrt{2\pi\hbar}}\int dp\,e^{i(p/\hbar)x}\,\frac{1}{\sqrt{2\pi\hbar}}\int dx'\,e^{-i(p/\hbar)x'}\,V(x') \\
&\qquad [\![\text{Note use of } x'\text{, not } x\text{, as dummy integration variable!}]\!] \\
=\;& \frac{1}{2\pi}\int dx'\,V(x')\int\frac{dp}{\hbar}\,e^{i(p/\hbar)(x-x')} \qquad [\![\text{Use analytic form of Dirac delta function…}]\!] \\
=\;& \int dx'\,V(x')\delta(x-x') \\
=\;& V(x).
\end{aligned}$$

Proof in the "bra-ket style":

$$\begin{aligned}
&\int dp\,\langle x|p\rangle\tilde{V}(p) \qquad [\![\text{Use definition (6.17b) …}]\!] \\
=\;& \int dp\,\langle x|p\rangle\int dx'\langle p|x'\rangle V(x')
\end{aligned}$$

[[Note use of x', not x, as dummy integration variable!]]

$$
\begin{aligned}
&= \int dx' \, \langle x| \int dp \, |p\rangle\langle p| \; |x'\rangle V(x') \qquad \text{[[Recognize the complete basis states...]]} \\
&= \int dx' \, \langle x|\hat{1}|x'\rangle V(x') \qquad \text{[[Recognize the orthogonal states...]]} \\
&= \int dx' \, \langle x|x'\rangle V(x') = \int dx' \, \delta(x - x') V(x') \\
&= V(x).
\end{aligned}
$$

d. Recall that

$$
\begin{aligned}
\langle p|\hat{V}|\psi(t)\rangle &= \frac{1}{\sqrt{2\pi\hbar}} \int dx \, e^{-i(p/\hbar)x} \, V(x)\psi(x;t) \\
V(x) &= \frac{1}{\sqrt{2\pi\hbar}} \int dp'' \, e^{i(p''/\hbar)x} \, \tilde{V}(p'') \\
\psi(x;t) &= \langle x|\psi(t)\rangle = \int dp' \, \langle x|p'\rangle\langle p'|\psi(t)\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dp' \, e^{i(p'/\hbar)x} \, \tilde{\psi}(p';t)
\end{aligned}
$$

so

$$
\begin{aligned}
\langle p|\hat{V}|\psi(t)\rangle &= \frac{1}{(\sqrt{2\pi\hbar})^3} \int dx \int dp' \int dp'' \, e^{-i(p/\hbar)x} e^{i(p'/\hbar)x} e^{i(p''/\hbar)x} \, \tilde{V}(p'')\tilde{\psi}(p';t) \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dp' \int dp'' \left[\frac{1}{2\pi} \int \frac{dx}{\hbar} \, e^{i(x/\hbar)(p'+p''-p)} \right] \tilde{V}(p'')\tilde{\psi}(p';t) \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dp' \int dp'' \left[\delta(p' + p'' - p) \right] \tilde{V}(p'')\tilde{\psi}(p';t) \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dp' \, \tilde{V}(p - p')\tilde{\psi}(p';t).
\end{aligned}
$$

e. Drawing all the pieces together,

$$
\frac{\partial\tilde{\psi}(p;t)}{\partial t} = -\frac{i}{\hbar} \left[\frac{p^2}{2m}\tilde{\psi}(p;t) + \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp' \, \tilde{V}(p - p')\tilde{\psi}(p';t) \right].
$$

One interesting point is that the Schrödinger equation is *local* in position space (the time rate of change of $\psi(x;t)$ at point x_0 depends only upon the value and curvature of $\psi(x;t)$ at that point) whereas the Schrödinger equation is *non-local* in momentum space (the time rate of change of $\tilde{\psi}(p;t)$ at momentum p_0 depends upon the values of $\tilde{\psi}(p;t)$ at all momenta from $-\infty$ to $+\infty$.)

[[*Grading:* This is a long and intricate problem, with a rich payoff. Students earn 2 points for each of the five parts.]]