Quantal recurrence in the infinite square well

a. Classical period:

$$E = \tfrac{1}{2}mv^2 \quad \text{so} \quad v = \sqrt{2E/m}$$

and

$$\text{distance} = \text{speed} \times \text{time},$$

so

$$\text{period} = \frac{\text{distance}}{\text{speed}} = \frac{2L}{\sqrt{2E/m}} = L\sqrt{\frac{2m}{E}}. \tag{1}$$

b. Quantal recurrence:
How does the initial wavefunction $\psi(x;0)$ change with time? Expanded the initial wavefunction into energy eigenfunctions $\eta_n(x)$:

$$\psi(x;0) = \sum_{n=1}^{\infty} c_n \eta_n(x). \tag{2}$$

This wavefunction evolves in time to

$$\psi(x;t) = \sum_{n=1}^{\infty} c_n e^{-iE_n t/\hbar} \eta_n(x), \tag{3}$$

where the eigenvalues are

$$E_n = \frac{\pi^2\hbar^2}{2ML^2} n^2 = E_1 n^2 \quad \text{for} \quad n = 1, 2, 3, \ldots . \tag{4}$$

The time-evolved wavefunction will equal the initial wavefunction whenever all of the phase factors $e^{-iE_n t/\hbar}$ are equal to one. That is, the revival occurs at a time T_{rev} where

$$\frac{E_n}{\hbar} T_{\text{rev}} = 2\pi \text{ (some integer)}$$

for all values of n. Using the eigenenergy result this becomes

$$\frac{E_1}{\hbar} T_{\text{rev}} n^2 = 2\pi \text{ (some integer)}$$

so the revival time is

$$T_{\text{rev}} = \frac{2\pi\hbar}{E_1} = \frac{h}{E_1} = \frac{4mL^2}{\pi\hbar}. \tag{5}$$

(Note that we solved this part knowing only the energy eigenvalues.)

c. What happens after one-half of this time has passed?
Evaluated at $t = T_{\text{rev}}/2$, equation (3) gives

$$\psi(x;T_{\text{rev}}/2) = \sum_{n=1}^{\infty} c_n e^{-iE_n T_{\text{rev}}/2\hbar} \eta_n(x). \tag{6}$$

But $T_{\text{rev}} = h/E_1$, so

$$\frac{E_n T_{\text{rev}}}{2\hbar} = \frac{E_1 T_{\text{rev}}}{2\hbar} n^2 = \pi n^2$$

and

$$\psi(x;T_{\text{rev}}/2) = \sum_{n=1}^{\infty} c_n e^{-i\pi n^2} \eta_n(x).$$

Now

$$e^{-i\pi n^2} = (-1)^{n^2} = (-1)^n$$

so

$$\psi(x;T_{\text{rev}}/2) = \sum_{n=1}^{\infty} c_n (-1)^n \eta_n(x). \tag{7}$$

But the energy eigenfunction $\eta_n(x)$ is even for n odd and odd for n even, so

$$(-1)^n \eta_n(x) = -\eta_n(-x)$$

whence

$$\psi(x;T_{\text{rev}}/2) = -\psi(-x;0). \tag{8}$$

That is: After half a revival time, the initial wavefunction is flipped from left to right and turned up-side down (that is, multiplied by the physically-irrelevant overall phase factor of -1). (Note that we solved this part knowing only the energy eigenvalues and the parity of the energy eigenfunctions.)

d. Numerical recurrence time:
If $m = 1$ kg and $L = 1$ m, then the recurrence time evaluates to 1.2×10^{34} s. In contrast, the age of the universe is about 13.82 billion years or 4.4×10^{17} s.

Does that mean a macroscopic particle in a macroscopic square well will require about 2.7×10^{16} times the age of the universe before it executes a single return to its starting point? We all know from common experience that that's false. The quantal recurrence time is much longer than the classical period because the classical period arrives when the position $x(t)$ and momentum $p(t)$ come back to their original values. The quantal recurrence time arrives when the mean position $\langle \hat{x} \rangle_t$ and mean momentum $\langle \hat{p} \rangle_t$ and the indeterminacy in position $(\Delta x)_t$ and the indeterminacy in momentum $(\Delta p)_t$ and the position-momentum correlation $\langle \hat{x}\hat{p} \rangle_t$ and the more elaborate correlation $\langle \hat{x}^3\hat{p}\hat{x}^2\hat{p}^4 \rangle_t$ and indeed the entire wavefunction come back to their original values. Because there are so many more conditions, of course it requires more time.

Grading:

2 points for part **a**.
2 points for equation (3).
2 more points for finishing off part **b**.
1 point for reaching equation (7).
1 more point for finishing off part **c**.
1 point for numerical value.
1 point for discussion of numerical value.