Electric fields and charges in elementary circuits

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In an effort to clarify the role of surface charges on the conductors of elementary electric circuits and the electric fields in the space around them, we present a quantitative analysis of (two-dimensional) circular current loops. It is also noted that, in general, lines of Poynting flux lie in the equipotential surfaces of quasistatic systems.

There is a peculiar discontinuity in the usual presentation of the first two major electrical topics in standard introductory physics courses. In electrostatics, our attention focuses explicitly on the electric charges residing on conductors and on the electric fields existing in the space external to the conductors. Then, in the next lecture (or chapter), we proceed to electric circuits and our attention focuses on batteries, resistors, hookup wire, and eventually capacitors. A perceptive freshman may see the apparent *non sequitur* and ask, "Don't we have to worry about charge distributions any more?" and "What are the electric fields in the vicinity of the circuit?" The student might even wonder, "How does a conduction electron in a crooked piece of wire, a long way from the battery, know enough to turn the corner?"

These issues are only rarely dealt with in textbooks, at either the introductory or intermediate level. The most complete textbook discussion known to this writer is that of Jefimenko.[1] Solutions to "Merzbacher's puzzle"[2] have already been discussed many times in the *American Journal of Physics* over the years. The fact that so authoritative a teacher raised the topic again suggests its subtlety and intransigence.[3]

I. THE SIMPLE ANSWER

The simple answer to our "perceptive freshman's" questions is that most of the time we *don't need to know* what the charge distributions and external fields are, which is fortunate because they are usually very difficult to calculate or to measure. Such charges and fields *do exist*—nature provides them automatically and they are essential for guiding the conduction current along the wires—but they do not figure explicitly in our quantitative analysis or practical construction of ordinary electric circuits. The charges and fields go hand in hand: if we can neglect one, we can neglect the other; if we insist on talking about one, we must include the other.

For instance, Kirchhoff's loop rule (suitably stated in terms of the sum of potential increments) applies to any mathematical path we might choose. But by choosing a path that coincides with the hardware of the circuit, we have simple means to bookkeep the increments in potential as we proceed around the loop (emf's, IR-drops, etc.). This is sufficient for a quantitative analysis of the behavior of the circuit. Infinitely many other "Kirchhoff-loop" equations exist for loops that do not coincide with the circuit but they do not allow us to solve for anything of interest.

For strictly dc circuits, the Kirchhoff point rule is rigorous. When the potentials are time varying, the point rule is no longer rigorous because of the time-varying charge on the surface of the hookup wire. Fortunately a great deal of electronic technology is at low enough frequencies that the numerical error in continuing to use the point rule is well below the threshold of detectability.

The reason not just freshmen but even professional electronics engineers do not want to talk about the charges distributed on electric circuits is that it is very difficult to determine these charges quantitatively. So one goes to great pains to avoid needing to determine them. This is usually done by restricting consideration to the "slowly varying" or "quasistatic" case, typically frequencies below a few megahertz. For higher frequencies, in later courses in high-frequency electronics and electromagnetic fields, the student learns about the importance of "stray capacitance" and the technology of transmission lines and waveguides, which were invented in the spirit of "when you can't fight'em, join'em." That is, in these latter cases the geometries are chosen in such a way that the charges and fields are well defined and calculatable.

As long as we stay in the "slowly varying" limit, not only can we use the Kirchhoff point rule but also, to the same level of precision, we can accept the well-known but nonobvious rule concerning capacitors that the charges on the two plates of any one capacitor always occur with equal magnitude and opposite sign. This rule follows from Gauss's law and the assumption that the only non-negligible electric field is in the small space between the capacitor plates.

The charges also are responsible for forces that one portion of a circuit exerts upon another.[4] Fortunately most practical circuits are rigid enough that these forces have negligible effect and we have no need to calculate them.

II. IMPORTANCE OF GEOMETRY

The reason that it is so hard to know the surface charges on the conductors of a circuit is that their distribution depends on the detailed geometry of the circuit itself and even of its surroundings.[5] For instance, we would have to specify exactly how the pieces of hookup wire are bent. And since most real-world circuits have rather complicated geometries, the mathematical difficulty of making this calculation is forbidding.

There are two useful things that can be done, however. One is to choose a very simple, idealized geometry that is tractable and study it as a test case. The other is to look for qualitative effects without trying to get quantitative.

The most obvious simple geometry is a very long, straight, resistive wire. This problem has been treated by a number of authors.[6–10] To make the problem well defined, the wire is assumed to be located normally between capacitor plates of infinite area, or part of a coaxial or parallel-wire circuit (the usual transmission-line geometries). The general conclusion is that the surface charge density is a *linear* function of the axial coordinate. The place where the charge density is locally zero depends upon details of the formulation of the problem.

With guidance from this one class of quantitative solutions, a number of authors have discussed qualitatively what makes the current flow,[11,12] what makes the current turn a corner,[13,14] and how crosstalk arises between "shielded" coaxial cables.[15]

III. THE CURRENT LOOP

The purpose of the present paper is to give a quantitative and pictorial treatment of two examples from a second class of idealized circuit geometries. The context of the discussion of elementary circuits in an introductory course and, indeed in all of "slowly varying" circuit theory, is not the long straight wire but rather the *loop*. In its primitive form it consists of a battery connected to a resistor with hookup wire, as in Fig. 1(a). Our aim is to see how far we can deal quantitatively with a system approaching this elementary prototype.[16,17]

The habit of drawing circuits schematically with square corners [Fig. 1(a)] is only an esthetic convention; real circuits rarely look like that. We can better model an actual physical circuit by supposing it to be laid out in a precisely circular geometry, as in Fig. 1(b). For present purposes, we may even think of the resistor as being distributed uniformly around the entire loop, as in Fig. 1(c). In both frames (b) and (c) of Fig. 1 we use the conventional zig-zag line to symbolize resistance; the actual resistor that these diagrams represent would be a nonwiggly resistive wire (nichrome, let us say) conforming to the circular geometry.

Even with these simplifications we still have a rather intractable problem. The limit as the radius of the wire goes to zero is not well defined. If we were to attempt to solve Laplace's equation for the electrostatic potential in the vicinity of the wire, we would have to deal with the boundary condition on the *toroidal* surface of the finite-size wire (even if we idealize the battery as of negligible size). In principle this problem can be solved using numerical methods such as relaxation,[18] but the boundary geometry and the three dimensionality are very awkward. The related magnetic problem must also be solved if we want to find the Poynting flux.[19]

Thus we are led to restrict our consideration to the analogous two-dimensional problem. We reinterpret frames (b) and (c) of Fig. 1 as representing cross sections of infinite cylinders. The battery is now in the shape of a line normal to the diagram at the left. A sheet current flows azimuthally around the cylindrical conducting sheet ("hookup wire" and resistor).[20] We shall see that this idealized geometry is quite tractable. Among other things, it is well behaved as the annular thickness of the conducting sheet goes to zero. The analysis is at the intermediate level; the resulting diagrams are easily understood at the introductory level.

IV. DISTRIBUTED RESISTOR

We consider the cylindrical geometry of Fig. 1(c), with a "line" battery driving current azimuthally in a uniform cylindrical resistive sheet (the zig-zag line for the resistor in the figure is only symbolic). We use conventional cylindrical coordinates r,θ,z coaxial with the cylinder. The radius of the cylinder is a; the annular thickness is negligible. The battery is located at $\theta = \pm\pi$; its terminals are at potentials $\pm V_0/2$. For some purposes it will be easier to work in Cartesian coordinates, $x = r\cos\theta$ and $y = r\sin\theta$.

In accordance with Ohm's law the potential at the conductor is

$$V(r=a;\theta) = V_0\theta/2\pi \quad (-\pi<\theta<\pi). \tag{1}$$

Using the well-known Fourier identity[21]

$$\theta = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\sin(k\theta)}{k} \quad (-\pi<\theta<\pi) \tag{2}$$

and the cylindrical harmonics for solutions independent of the z coordinate,[22] we can immediately express the potential inside and outside in the form of the infinite series

$$V(r<a;\theta) = -\frac{V_0}{\pi}\sum_{k=1}^{\infty}\frac{(-r/a)^k\sin(k\theta)}{k}, \tag{3}$$

$$V(r>a;\theta) = -\frac{V_0}{\pi}\sum_{k=1}^{\infty}\frac{(-a/r)^k\sin(k\theta)}{k}. \tag{4}$$

These series can be put in the closed forms[23]

$$V(r<a;\theta) = (V_0/\pi)\tan^{-1}[r\sin\theta/(a+r\cos\theta)]$$
$$= (V_0/\pi)\tan^{-1}[y/(a+x)] = (V_0/\pi)\phi, \tag{5}$$

$$V(r>a;\theta) = (V_0/\pi)\tan^{-1}[a\sin\theta/(r+a\cos\theta)]$$
$$= (V_0/\pi)\tan^{-1}[ay/(x^2+ax+y^2)], \tag{6}$$

where ϕ in Eq. (5) is the polar angle for an origin at the

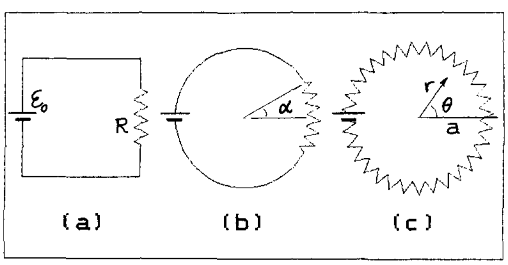

Fig. 1. The elementary current loop: (a) schematic diagram; (b) circular loop with localized ("lumped") resistor of angular size 2α; (c) circular loop with distributed resistor, showing polar coordinate system. The loop radius is a.

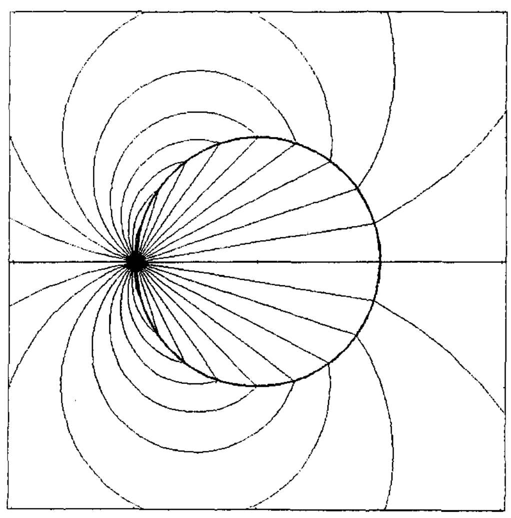

Fig. 2. Equipotentials for (two-dimensional) current loop with distributed resistance. The battery, of negligible size, is at the dot at the left. The lines inside the loop can also be interpreted as lines of Poynting flux.

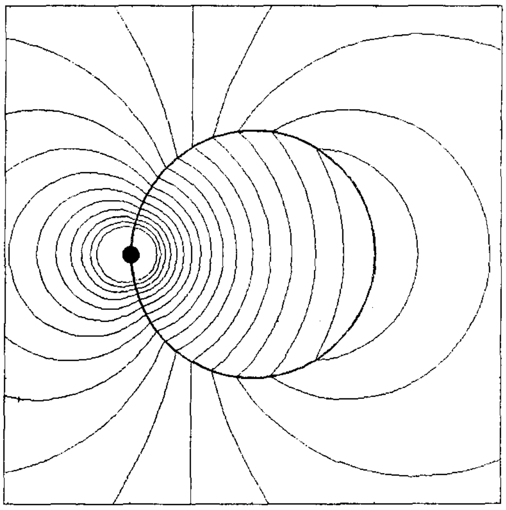

Fig. 3. Electric lines of force for current loop with distributed resistance.

battery. Figure 2 shows a representative family of equipotentials.

Inside the loop, the equipotentials are straight lines (strictly, planes extending in the z dimension), as indicated in the final form of Eq. (5). Outside, the equipotentials are circular arcs (cylindrical surfaces), given by

$$(x + \tfrac{1}{2}a)^2 + [y - \tfrac{1}{2}a \cot(\pi V/V_0)]^2 = [\tfrac{1}{2}a \csc(\pi V/V_0)]^2. \quad (7)$$

All outer equipotentials pass through the battery at $x = -a$, $y = 0$, and extrapolate through the origin $x = y = 0$. Thus they are symmetric about the vertical plane at $x = -\frac{1}{2}a$.

The electric field at any point can now be found by calculating the gradient of the potential. The lines of force are shown in Fig. 3.[20] Note that the lines of force leave the conducting surface diagonally as explained qualitatively by Parker[12] and many others.

Inside the loop, the lines of force are circular arcs centered on the battery, given by

$$(x + a)^2 + y^2 = (2^{1-N}a)^2, \quad (8)$$

where N is a parameter identifying a particular field line.[24] The magnitude of the field inside is[20]

$$|E|_{r<a} = (V_0/\pi)\,(1/\rho), \quad (9)$$

where ρ is the polar radius measured from the battery.

Outside the loop, the lines of force are again circular arcs, given by[25]

$$[x - a/(2^{2-2N} - 1)]^2 + y^2 = [2^{1-N}a/(2^{2-2N} - 1)]^2. \quad (10)$$

The magnitude of the field outside is

$$|E|_{r>a} = (V_0/\pi)(a/r\rho), \quad (11)$$

where r and ρ are the polar radii measured from the center (axis) and from the battery, respectively. At large distances, the system has the field of a (two-dimensional) dipole located at $x = -\frac{1}{2}a$, $y = 0$. The field just outside is similar, but not identical, to that of a finite (two-dimensional) dipole.[26]

V. SURFACE CHARGE AND POYNTING VECTOR

From the radial component of the field as $r \to a$ we can find the surface charge density,

$$\sigma(\theta) = (\pm)\epsilon_0 E_r = (\epsilon_0 V_0/2\pi a)\tan(\tfrac{1}{2}\theta)$$
$$= (\epsilon_0 V_0/2\pi a)\tan\phi, \quad (12)$$

where again ϕ is the polar angle with respect to an origin at the battery ($-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$). This same charge density exists on both the inner and outer surfaces of the resistive conductor. It is no longer a linear function of the length coordinate (perimeter) of the resistor, as it was for the long straight conductor,[6–10] but a tangent function that increases nonlinearly towards the battery.

The magnetic field of this example is the elementary case of an infinite solenoid: the magnitude of the field is uniform inside and zero outside, and its direction is (inwardly) normal to the figure. The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ lies in the plane of the figure, orthogonal to the **E**-field lines of force, and thus coincides with the equipotentials.[27] That is, we can interpret the equipotential lines inside the loop in Fig. 2 as being lines of Poynting flux. The vector sense is from the battery to the resistor with the physical interpretation that the Poynting flux represents the path of transport of energy from the interior of the battery as source to the Joule (I^2R) sink in the resistor.[28] It is easy to see from physical arguments that, since the magnetic field is constant, a family of equipotentials chosen to represent equal steps of potential (as in Fig. 2) also represents equal increments of Poynting flux, in accordance with the convention that the density of flux lines is proportional to the local magnitude of the Poynting vector.

Figures 2 and 3, and Eq. (12), are quantitatively correct only for the two-dimensional (cylindrical) case. Nevertheless, they are not far from what one would expect for the three-dimensional case of a loop consisting of a "point" battery and a thin resistive wire. In that case, of course, the magnetic field outside the loop is nonzero and of reversed sense. Accordingly, the equipotentials *outside* the loop, in the analog of Fig. 2 for the three-dimensional wire, would also coincide with lines of Poynting flux. That is, energy is fed to elements of the wire from all sides, including normal to the plane of the figure, in contrast to the two-dimensional case where the sheet resistor is fed energy only from the inside.

The charges on the surface of the wire provide two types of electric field. The charges provide the field *inside* the wire that drives the conduction current according to Ohm's law. Simultaneously the charges provide a field *outside* the wire that creates a Poynting flux. By means of this latter field, the charges enable the wire to be a guide (in the sense of a railroad track) for electromagnetic energy flowing in the space around the wire. Intuitively one might prefer the notion that electromagnetic energy is transported by the current, inside the wires. It takes some effort to convince oneself (and one's students) that this is not the case and that in fact the energy flows in the space *outside* the wire.[29]

It is easy to generalize the coincidence of equipotentials and lines of Poynting flux for "slowly varying" systems in which Faraday electric fields can be neglected. The electric field at a point, being the gradient of the potential, is perpendicular to the equipotential surface passing through the point. The Poynting vector, proportional to the vector product $\mathbf{E} \times \mathbf{B}$, is necessarily perpendicular to the electric field. Therefore the Poynting vector at the point, and its extension into a continuous flux line, must lie in that equipotential surface.[27] In a two-dimensional diagram of a particular three-dimensional electric system, the equipotentials generally appear as lines, being the intersection of a family of equipotential surfaces with the plane of the diagram. These equipotential lines may or may not be lines of Poynting flux depending upon the absence or presence of a component of the Poynting vector normal to the diagram. In many cases of interest one can use symmetry arguments to choose a diagram plane that is tangent to the Poynting lines, which then coincide with the equipotential lines. An example is the plane of any multimesh planar circuit.

VI. LUMPED RESISTOR

We now return to the case of a "lumped" resistor, as in Fig. 1(b), and again interpret the figure as two-dimensional (a "line" battery; currents flowing azimuthally over the surface of a cylinder). Let the resistive conductor be limited to the angular portion $-\alpha < \theta < \alpha$, using the same coordinate system as for the distributed-resistor loop [Fig. 1(c)]. The rest of the circuit, the "hookup wires" (perfectly conducting cylindrical surfaces), are equipotential with the terminals of the battery.

It is not difficult to show that the potential at the loop ($r = a$), replacing Eq. (1), is given by the Fourier series

$$V(r=a;\theta) = \sum_{k=1}^{\infty} A_k \sin(k\theta), \tag{13}$$

where the coefficients are

$$A_k = (V_0/\pi)\left(\int_0^{\alpha} (\theta/\alpha)\sin(k\theta)d\theta + \int_{\alpha}^{\pi} \sin(k\theta)d\theta\right)$$
$$= (V_0/\pi)[(-1)^{k-1}/k + \sin(k\alpha)/k^2\alpha]. \tag{14}$$

The first term of the final form of Eq. (14) is exactly that for the distributed resistor ($\alpha \to \pi$), for which we have the closed-form formulas of Eqs. (5) and (6). Thus we obtain

$$V(r<a;\theta) = (V_0/\pi)\Big(\tan^{-1}[r\sin\theta/(a + r\cos\theta)]$$
$$+ \sum_k (r/a)^k [\sin(k\alpha)/k^2\alpha]\sin(k\theta)\Big), \tag{15}$$

$$V(r>a;\theta) = (V_0/\pi)\Big(\tan^{-1}[a\sin\theta/(r + a\cos\theta)]$$
$$+ \sum_k (a/r)^k [\sin(k\alpha)/k^2\alpha]\sin(k\theta)\Big). \tag{16}$$

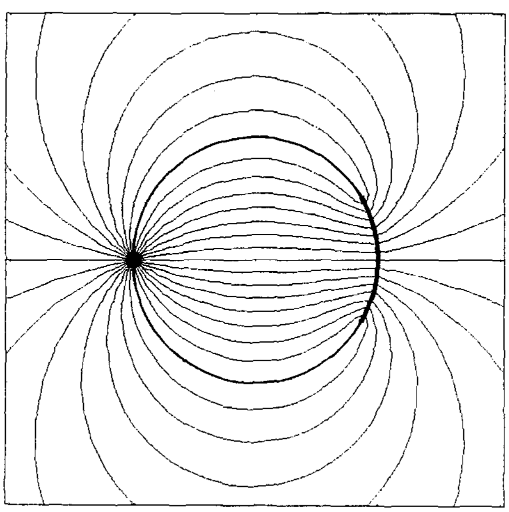

Fig. 4. Equipotentials for (two-dimensional) current loop with "lumped" resistor ($\alpha = 30°$). The heavier curve represents the resistive portion of the circuit. The equipotential curves inside the loop can also be interpreted as lines of Poynting flux.

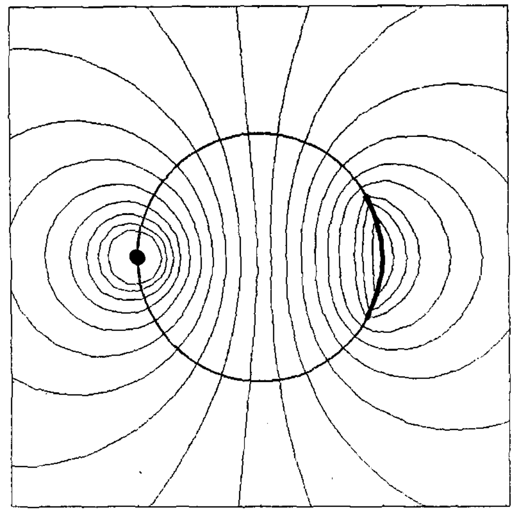

Fig. 5. Electric lines of force for current loop with "lumped" resistor.

The remaining infinite series converge as k^{-2} and can be truncated after a reasonable number of terms. We evaluated these formulas numerically for a 31×31 grid of points ($0 \leqslant r \leqslant 3a, 0 \leqslant \theta \leqslant \pi$) and then generated Figs. 4 and 5 using numerical potential- and gradient-tracing computer routines.[30] In this case the origin of the farfield dipole is only slightly to the left of the center (axis).

If we reinterpret the equipotentials of Fig. 4 as lines of Poynting flux, we again see energy flowing from the battery as source to the elements of the resistor as sink, but not to the hookup wire.

The surface charge density (on both inner and outer surfaces), found from Eqs. (15) and (16), is[31]

$$\sigma(\theta) = (\epsilon_0 V_0/2\pi a)\Big(\tan(\tfrac{1}{2}\theta)$$
$$+ \sum_k 2[\sin(k\alpha)/k\alpha]\sin(k\theta)\Big)$$
$$= (\epsilon_0 V_0/2\pi a)[\tan(\tfrac{1}{2}\theta)$$
$$+ (1/\alpha)\ln|\sin\tfrac{1}{2}(\theta + \alpha)/\sin\tfrac{1}{2}(\theta - \alpha)|]. \tag{17}$$

To the tangent function of Eq. (12) is added a term with logarithmic singularities at $\theta = \pm\alpha$, as shown in Fig. 6. Additional charge is required to provide the increased field in the resistor and to remove any field parallel to the perfectly conducting hookup wire. Note in Fig. 5 that the lines of force leave the resistor obliquely (there are both parallel and normal components of field), whereas they leave the

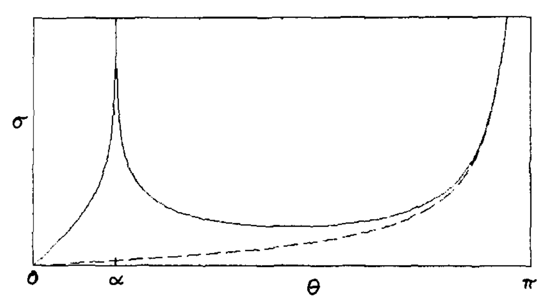

Fig. 6. Surface charge density from center of resistor ($\theta = 0$) to upper terminal of the battery ($\theta = \pi$); dashed curve for distributed resistor [Eq. (12)], solid curve for lumped resistor [Eq. (17)].

hookup wire at right angles (there is only a normal component). When we recognize that a real circuit must have finite thickness, we see that in addition to the surface charges on the exterior there will be surface charges in the *interior* of the conductor at the interfaces between the portions of different resistivity.[14]

VII. CONCLUSION

Although our calculations are only for the two-dimensional (cylindrical) case, the results give considerable insight into the charge distributions and configurations of potential, electric field, and Poynting flux in elementary current loops in the three-dimensional real world. In particular, we have emphasized the role of the Poynting flux as carrying energy from the battery as source to elements of the resistor as sink. The external electric field produced by surface charges on the wires is the mechanism by which the Poynting flux is steered or guided by the wires.

In somewhat more complicated cases, such as plane multimesh circuits, one can sketch approximate equipotentials in the spaces inside and outside the meshes in such a way as to be consistent with the boundary conditions that are given by the potential distribution in the circuit elements, which in turn is obtained from the Kirchhoff analysis. Then these equipotential lines can be reinterpreted as lines of Poynting flux, or an orthogonal set of electric field lines can be sketched in. If your spatial imagination is good, you can even conjure up mental images of systems with three-dimensional equipotential surfaces, in which lie the Poynting flux lines connecting batteries to resistors.

ACKNOWLEDGMENT

This work was supported by Swarthmore College.

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[5]For a specific example, see Ref. 1, Fig. 9.11.

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[21]H. B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, New York, 1961), Formula 416.07.

[22]See, for instance, E. M. Pugh and E. W. Pugh, *Principles of Electricity and Magnetism* (Addison-Wesley, Reading, MA, 1970), 2nd ed., pp. 100–104.

[23]Calculate the radial component of the electric field, which can then be summed using Formula 417.4 of Ref. 21. Integrate to regain the potential using Formula 160.01.

[24]The choice of field lines plotted in Fig. 3 follows the usual convention of constant incremental flux between adjacent lines; accordingly the density of field lines is proportional to the local magnitude of the field. To accomplish this we label lines of force by the flux parameter

$$N = \int_a^x E_y(x;y=0)dx \Big/ \int_a^0 E_y(x;y=0)dx,$$

which measures the electric flux (per unit length of the cylinder) crossing the horizontal diameter ($y=0$) between its right end at $x=a$ and a given field line crossing in the domain $-a<x<a$. Thus $N=1$ identifies the line crossing at $x=0$, and $N\to\infty$ for lines near the battery as $x\to -a$. In Fig. 3, $\Delta N = 0.2$.

[25]The definition of the parameter N is identical to that given in Ref. 24 but with reversed sign. The domain is now $|x|>a$. Thus $N=0$ identifies the line that crosses the midplane at $x=a$, just outside the loop at the right; $N=1$ gives a vertical straight line at $x=-\frac{1}{2}a$, extending to infinity; and $N\to\infty$ for lines near the battery as $x\to -a$ from the left.

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[30]The writer will be glad to provide documentation on these routines upon request.

[31]Reference 21, Formula 603.2.

PROBLEM: TRANSFER ORBIT TO COUNTER EARTH

Science fiction writers, such as John Norman writing in *Chronicles of Counter Earth*, have described a sister planet which shares the same orbit with the Earth. When the Earth is at perihelion, Counter Earth is at aphelion. How could earthlings launch a spacecraft to explore Counter Earth? The problem is to find the most efficient transfer orbit—an elliptical orbit about the sun from Earth to Counter Earth. (Solution is on page 559.)