Alfven's theorem

Griffiths, ***Electrodynamics*****, fourth edition, problem 7.63**

(a) We have $\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$, but $\sigma \to \infty$ and $\vec{J}$ is finite, so $\vec{E} + \vec{v} \times \vec{B} \to 0$ or, to good approximation,

$$\vec{E} = -\vec{v} \times \vec{B}.$$

Thus Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

implies

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}).$$

(b) The sum of surfaces $\mathcal{S}'$ plus $\mathcal{R}$ plus $\mathcal{S}$ constitute a closed surface. Surfaces $\mathcal{S}'$ and $\mathcal{R}$ are oriented with normal vectors pointing outward, while surface $\mathcal{S}$ is oriented with normal vectors pointing inward (as suggested in the figure). Because

$$\oint \vec{B} \cdot \hat{n}\, dA = 0$$

for any closed surface, we have

$$\int_{\mathcal{S}'} \vec{B} \cdot \hat{n}\, dA + \int_{\mathcal{R}} \vec{B} \cdot \hat{n}\, dA = \int_{\mathcal{S}} \vec{B} \cdot \hat{n}\, dA.$$

Meanwhile, using the method at the bottom of page 307 and top of page 308 (and an outward normal vector),

$$\int_{\mathcal{R}} \vec{B} \cdot (\hat{n}\, dA) = \int_{\mathcal{P}} \vec{B} \cdot (d\vec{\ell} \times \vec{v}\, dt) = dt \int_{\mathcal{P}} \vec{B} \cdot (d\vec{\ell} \times \vec{v}) = dt \int_{\mathcal{P}} d\vec{\ell} \cdot (\vec{v} \times \vec{B}) = dt \int_{\mathcal{P}} (\vec{v} \times \vec{B}) \cdot d\vec{\ell}.$$

Whence

$$\frac{d\Phi}{dt} = \int_{\mathcal{S}} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, dA - \int_{\mathcal{P}} (\vec{v} \times \vec{B}) \cdot d\vec{\ell}.$$

According to Stokes, for any vector field $\vec{F}$

$$\int_{\mathcal{P}} \vec{F} \cdot d\vec{\ell} = \int_{\mathcal{S}} (\nabla \times \vec{F}) \cdot \hat{n}\, dA.$$

Apply Stokes to the field $\vec{F} = \vec{v} \times \vec{B}$ giving

$$\int_{\mathcal{P}} (\vec{v} \times \vec{B}) \cdot d\vec{\ell} = \int_{\mathcal{S}} (\nabla \times (\vec{v} \times \vec{B})) \cdot \hat{n}\, dA.$$

so

$$\frac{d\Phi}{dt} = \int_{\mathcal{S}} \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, dA - \int_{\mathcal{S}} (\nabla \times (\vec{v} \times \vec{B})) \cdot \hat{n}\, dA = \int_{\mathcal{S}} \left(\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) \right) \cdot \hat{n}\, dA = 0.$$

QED.