Addition of Angular Momenta

Griffiths problem 4.34

(a) Remember that $\hat{S}_- = \hat{S}_-^{(A)} + \hat{S}_-^{(B)}$ and that

$$\hat{S}_- |s, m\rangle = \hbar\sqrt{s(s+1) - m(m-1)}\,|s, m-1\rangle.$$

Thus

$$\begin{aligned}
\hat{S}_- \uparrow &= \hbar\sqrt{\tfrac{1}{2}(\tfrac{3}{2}) - \tfrac{1}{2}(-\tfrac{1}{2})}\downarrow = \hbar \downarrow, \\
\hat{S}_- |1, 0\rangle &= \hbar\sqrt{1(2) - 0(-1)}\,|1, -1\rangle = \hbar\sqrt{2}\,|1, -1\rangle.
\end{aligned}$$

So

$$\begin{aligned}
|1, 0\rangle &= \tfrac{1}{\sqrt{2}}[\uparrow\downarrow + \downarrow\uparrow] \\
\hat{S}_- |1, 0\rangle &= [\hat{S}_-^{(A)} + \hat{S}_-^{(B)}]\tfrac{1}{\sqrt{2}}[\uparrow\downarrow + \downarrow\uparrow] \\
&= \tfrac{1}{\sqrt{2}}[(\hat{S}_-^{(A)} \uparrow) \downarrow + \uparrow (\hat{S}_-^{(B)} \downarrow) + (\hat{S}_-^{(A)} \downarrow) \uparrow + \downarrow (\hat{S}_-^{(B)} \uparrow)] \\
&= \tfrac{1}{\sqrt{2}}[(\hbar \downarrow) \downarrow + \uparrow (0) + (0) \uparrow + \downarrow (\hbar \downarrow)] \\
&= \sqrt{2}\,\hbar \downarrow\downarrow,
\end{aligned}$$

which agrees precisely with the result of $\hat{S}_- |1, 0\rangle$.

(b) Lowering:

$$\begin{aligned}
|0, 0\rangle &= \tfrac{1}{\sqrt{2}}[\uparrow\downarrow - \downarrow\uparrow] \\
\hat{S}_- |0, 0\rangle &= [\hat{S}_-^{(A)} + \hat{S}_-^{(B)}]\tfrac{1}{\sqrt{2}}[\uparrow\downarrow - \downarrow\uparrow] \\
&= \tfrac{1}{\sqrt{2}}[(\hat{S}_-^{(A)} \uparrow) \downarrow + \uparrow (\hat{S}_-^{(B)} \downarrow) - (\hat{S}_-^{(A)} \downarrow) \uparrow - \downarrow (\hat{S}_-^{(B)} \uparrow)] \\
&= \tfrac{1}{\sqrt{2}}[(\hbar \downarrow) \downarrow + \uparrow (0) - (0) \uparrow - \downarrow (\hbar \downarrow)] \\
&= 0.
\end{aligned}$$

Raising:

$$\begin{aligned}
|0, 0\rangle &= \tfrac{1}{\sqrt{2}}[\uparrow\downarrow - \downarrow\uparrow] \\
\hat{S}_+ |0, 0\rangle &= [\hat{S}_+^{(A)} + \hat{S}_+^{(B)}]\tfrac{1}{\sqrt{2}}[\uparrow\downarrow - \downarrow\uparrow] \\
&= \tfrac{1}{\sqrt{2}}[(\hat{S}_+^{(A)} \uparrow) \downarrow + \uparrow (\hat{S}_+^{(B)} \downarrow) - (\hat{S}_+^{(A)} \downarrow) \uparrow - \downarrow (\hat{S}_+^{(B)} \uparrow)] \\
&= \tfrac{1}{\sqrt{2}}[(0) \downarrow + \uparrow (\hbar \uparrow) - (\hbar \uparrow) \uparrow - \downarrow (0)] \\
&= 0.
\end{aligned}$$

(c) Follow the reasoning on page 186:

$$\hat{S}^2 = (\hat{S}^{(A)})^2 + (\hat{S}^{(B)})^2 + 2\hat{\mathbf{S}}^{(A)} \cdot \hat{\mathbf{S}}^{(B)}.$$

First, apply this operator to $|1,+1\rangle = \uparrow\uparrow$. We work piece-by-piece:

$$\begin{aligned}
\hat{\mathbf{S}}^{(A)} \cdot \hat{\mathbf{S}}^{(B)}(\uparrow\uparrow) &= (\hat{S}_x^{(A)} \uparrow)(\hat{S}_x^{(B)} \uparrow) + (\hat{S}_y^{(A)} \uparrow)(\hat{S}_y^{(B)} \uparrow) + (\hat{S}_z^{(A)} \uparrow)(\hat{S}_z^{(B)} \uparrow) \\
&= \left(\frac{\hbar}{2} \downarrow\right)\left(\frac{\hbar}{2} \downarrow\right) + \left(\frac{i\hbar}{2} \downarrow\right)\left(\frac{i\hbar}{2} \downarrow\right) + \left(\frac{\hbar}{2} \uparrow\right)\left(\frac{\hbar}{2} \uparrow\right) \\
&= \tfrac{1}{4}\hbar^2 \uparrow\uparrow .
\end{aligned}$$

Meanwhile

$$(\hat{S}^{(A)})^2 \uparrow\uparrow = [(\hat{S}^{(A)})^2 \uparrow] \uparrow = [\hbar^2 \tfrac{1}{2}(\tfrac{1}{2}+1) \uparrow] \uparrow = \tfrac{3}{4}\hbar^2 \uparrow\uparrow$$

and

$$(\hat{S}^{(B)})^2 \uparrow\uparrow = \uparrow [(\hat{S}^{(B)})^2 \uparrow] = \uparrow [\hbar^2 \tfrac{1}{2}(\tfrac{1}{2}+1) \uparrow] = \tfrac{3}{4}\hbar^2 \uparrow\uparrow .$$

Thus

$$\hat{S}^2(\uparrow\uparrow) = \tfrac{3}{4}\hbar^2 \uparrow\uparrow + \tfrac{3}{4}\hbar^2 \uparrow\uparrow + 2(\tfrac{1}{4}\hbar^2 \uparrow\uparrow) = 2\hbar^2 \uparrow\uparrow = \hbar^2(1)(1+1) \uparrow\uparrow$$

so the value of s associated with $\uparrow\uparrow$ is $s = 1$. Spin one!

Second, apply this operator to $|1,-1\rangle = \downarrow\downarrow$. Again working piece-by-piece, we have

$$\begin{aligned}
\hat{\mathbf{S}}^{(A)} \cdot \hat{\mathbf{S}}^{(B)}(\downarrow\downarrow) &= (\hat{S}_x^{(A)} \downarrow)(\hat{S}_x^{(B)} \downarrow) + (\hat{S}_y^{(A)} \downarrow)(\hat{S}_y^{(B)} \downarrow) + (\hat{S}_z^{(A)} \downarrow)(\hat{S}_z^{(B)} \downarrow) \\
&= \left(\frac{\hbar}{2} \uparrow\right)\left(\frac{\hbar}{2} \uparrow\right) + \left(-\frac{i\hbar}{2} \uparrow\right)\left(-\frac{i\hbar}{2} \uparrow\right) + \left(\frac{\hbar}{2} \downarrow\right)\left(\frac{\hbar}{2} \downarrow\right) \\
&= \tfrac{1}{4}\hbar^2 \downarrow\downarrow .
\end{aligned}$$

The remaining equations are exactly parallel to the $\uparrow\uparrow$ case, except that every $\uparrow$ is replaced by a $\downarrow$:

$$\begin{aligned}
(\hat{S}^{(A)})^2 \downarrow\downarrow &= [(\hat{S}^{(A)})^2 \downarrow] \downarrow = [\hbar^2 \tfrac{1}{2}(\tfrac{1}{2}+1) \downarrow] \downarrow = \tfrac{3}{4}\hbar^2 \downarrow\downarrow \\
(\hat{S}^{(B)})^2 \downarrow\downarrow &= \downarrow [(\hat{S}^{(B)})^2 \downarrow] = \downarrow [\hbar^2 \tfrac{1}{2}(\tfrac{1}{2}+1) \downarrow] = \tfrac{3}{4}\hbar^2 \downarrow\downarrow .
\end{aligned}$$

Thus

$$\hat{S}^2(\downarrow\downarrow) = \tfrac{3}{4}\hbar^2 \downarrow\downarrow + \tfrac{3}{4}\hbar^2 \downarrow\downarrow + 2(\tfrac{1}{4}\hbar^2 \downarrow\downarrow) = 2\hbar^2 \downarrow\downarrow = \hbar^2(1)(1+1) \downarrow\downarrow$$

so the value of s associated with $\downarrow\downarrow$ is $s = 1$. Spin one again!

Griffiths problem 4.35

(b) Meson spin values: Spins will range from $\frac{1}{2} + \frac{1}{2} = 1$ to $\frac{1}{2} - \frac{1}{2} = 0$ by integer steps, so they will be 1 or 0.

(a) Baryon spin values: Add two spins to get 1 or 0, as above. Add spin 1 and spin $\frac{1}{2}$ to get spin $\frac{3}{2}$ or spin $\frac{1}{2}$, then add spin 0 and spin $\frac{1}{2}$ to get spin $\frac{1}{2}$. So baryons have spin $\frac{3}{2}$ or spin $\frac{1}{2}$.

Griffiths problem 4.36

(a) We are adding spin 1 and spin 2 to make a total angular momentum state $|j, m_j\rangle = |3, +1\rangle$. What sorts of $|2, m_A\rangle|1, m_B\rangle$ states must we add together to build such a "J-type" state? Simple z-component counting tells us that $|3, +1\rangle$ can only be constructed from a combination of $|2, 0\rangle|1, +1\rangle$, $|2, +1\rangle|1, 0\rangle$, and $|2, +2\rangle|1, -1\rangle$. The actual recipie for adding these together is given through the Clebsch-Gordon table on page 188: Look within the table 2×1 to find the column "3 // +1". The meaning of this column is that

$$|3, +1\rangle = \sqrt{\frac{6}{15}}\,|2, 0\rangle|1, +1\rangle + \sqrt{\frac{8}{15}}\,|2, +1\rangle|1, 0\rangle + \sqrt{\frac{1}{15}}\,|2, +2\rangle|1, -1\rangle.$$

So if I measure m_A, the possible results and corresponding probabilities are

result	probability
0	6/15
+1	8/15
+2	1/15

(b) We need to add $\ell = 1$, $m_\ell = 0$ plus $s = \frac{1}{2}$, $m_s = -\frac{1}{2}$. Using the "$\ell + s$ to $|\ell - s|$" rule, these two angular momenta can sum to $j = \frac{3}{2}$ (in which case $|j, m_j\rangle = |\frac{3}{2}, -\frac{1}{2}\rangle$) or to $j = \frac{1}{2}$ (in which case $|j, m_j\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$). This gives us the possible results for j.

To find the probabilites of the two possible results, go to the Clebsch-Gordon table on page 188. Look at the $1 \times 1/2$ table, and go to the first row on the third block — the one labeled "0 // −1/2". The meaning of this row is (compare [4.186])

$$|1, 0\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{\frac{2}{3}}\,|\tfrac{3}{2}, -\tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|\tfrac{1}{2}, -\tfrac{1}{2}\rangle.$$

Thus the possible values of angular momentum squared, and their associated probabilities, are

j	$\hbar^2 j(j+1)$	probability
3/2	$(15/4)\hbar^2$	2/3
1/2	$(3/4)\hbar^2$	1/3