Solution of the RC circuit differential equation

$$\text{ODE:} \qquad \frac{dq(t)}{dt} + \frac{q(t)}{RC} = \frac{\mathcal{E}_m}{R}\sin(\omega t) \tag{1}$$

$$\text{ansatz:} \qquad q(t) = Q\cos(\omega t - \phi) \tag{2}$$

where the Q and ϕ are adjustable parameters that will depend (in ways to be uncovered) upon R, C, ω, and $\mathcal{E}_m$.

Plug the ansatz into the ODE, giving

$$-Q\omega\sin(\omega t - \phi) + \frac{Q}{RC}\cos(\omega t - \phi) = \frac{\mathcal{E}_m}{R}\sin(\omega t). \tag{3}$$

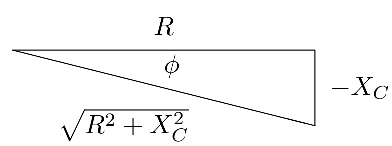
If this equation is to hold for all times, it must hold for the time $t = 0$:

$$-Q\omega\sin(-\phi) + \frac{Q}{RC}\cos(-\phi) = 0 \tag{4}$$

$$\tan\phi = -\frac{1}{\omega RC} \tag{5}$$

Define $X_C \equiv \dfrac{1}{\omega C}$ and call it the "reactance" or "impedance" of the capacitor (dimensions: ohm). Then

$$\tan\phi = -\frac{X_C}{R}. \tag{6}$$



If the ansatz is to hold for all times, then it must also hold for the time when $\omega t = \phi$, at which time equation (3) becomes

$$\frac{Q}{RC} = \frac{\mathcal{E}_m}{R}\sin\phi \tag{7}$$

$$Q = \mathcal{E}_m C\sin\phi = \mathcal{E}_m C\left(-\frac{X_C}{\sqrt{R^2 + X_C^2}}\right) \tag{8}$$

So we've shown that if ansatz (2) is a solution to ODE (1), then the parameters Q and ϕ must take on the values given in equations (6) and (8). But we've not yet shown that ansatz (2) is *really* a solution. To do so, plug equations (6) and (8) into (3)

$$\begin{aligned} -\omega\mathcal{E}_m C \sin\phi \sin(\omega t-\phi) + \frac{\mathcal{E}_m C \sin\phi}{RC}\cos(\omega t-\phi) &= \frac{\mathcal{E}_m}{R}\sin\omega t \qquad (9) \\ -\omega RC \sin\phi \sin(\omega t-\phi) + \sin\phi\cos(\omega t-\phi) &= \sin\omega t \qquad (10) \end{aligned}$$

and use the dreaded (at least, dreaded by me) sum angle formulas

$$\begin{aligned} \sin(\omega t-\phi) &= \sin\omega t\cos\phi - \cos\omega t\sin\phi \qquad (11) \\ \cos(\omega t-\phi) &= \cos\omega t\cos\phi + \sin\omega t\sin\phi. \qquad (12) \end{aligned}$$

This gives (using $\omega RC = R/X_C$)

$$[-\frac{R}{X_C}\sin\phi\cos\phi + \sin^2\phi]\sin\omega t + [\frac{R}{X_C}\sin^2\phi + \sin\phi\cos\phi]\cos\omega t = \sin\omega t. \qquad (13)$$

But from equation (6)

$$\begin{aligned} \sin\phi = -\frac{X_C}{\sqrt{R^2+X_C^2}} \quad &; \quad \cos\phi = \frac{R}{\sqrt{R^2+X_C^2}} \qquad (14) \\ \sin^2\phi = \frac{X_C^2}{R^2+X_C^2} \quad &; \quad \sin\phi\cos\phi = -\frac{RX_C}{R^2+X_C^2} \qquad (15) \end{aligned}$$

whence (13) becomes

$$[1]\sin\omega t + [0]\cos\omega t = \sin\omega t. \qquad (16)$$

Which is manifestly true!