Electric Potential Energy

Find the electric potential energy of three charges, q_1, q_2, and g_3, separated by the distances r_{12}, r_{13}, and r_{23}. (Note: if there were four charges, there would be six distances, if there were N charges there would be $N(N-1)/2$ distances.)

Initial configuration: The three charges all all far away from each other. By definition, the electric potential energy is $U^{(e)} = 0$.

Stage I: Move q_1 to its final position:

$$\begin{aligned} \Delta U^{(e)} &= -\int \vec{F}^{(e)}_{\text{on }1} \cdot d\vec{\ell}_1 \\ &= 0. \end{aligned}$$

So

$$\begin{aligned} U^{(e)}_{\text{at end of stage I}} &= U^{(e)}_{\text{at start of stage I}} + \Delta U^{(e)} \\ &= 0. \end{aligned}$$

Stage II: Move q_2 to its final position:

$$\begin{aligned} \Delta U^{(e)} &= -\int \vec{F}^{(e)}_{\text{on }2} \cdot d\vec{\ell}_2 \qquad [\![\ldots \text{place origin on top of } q_1 \ldots]\!] \\ &= -\int \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_2^2} \hat{r}_2 \cdot d\vec{\ell}_2 \\ &= +\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}. \end{aligned}$$

So

$$\begin{aligned} U^{(e)}_{\text{at end of stage II}} &= U^{(e)}_{\text{at start of stage II}} + \Delta U^{(e)} \\ &= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}. \end{aligned}$$

Stage III: Move q_3 to its final position:

$$\begin{aligned} \Delta U^{(e)} &= -\int \vec{F}^{(e)}_{\text{on }3} \cdot d\vec{\ell}_3 \\ &= -\int \vec{F}^{(e)}_{\text{on 3 by 1}} \cdot d\vec{\ell}_3 - \int \vec{F}^{(e)}_{\text{on 3 by 2}} \cdot d\vec{\ell}_3 \\ &= +\frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{r_{23}}. \end{aligned}$$

So

$$\begin{aligned} U^{(e)}_{\text{at end of stage III}} &= U^{(e)}_{\text{at start of stage III}} + \Delta U^{(e)} \\ &= \frac{1}{4\pi\epsilon_0} \left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right]. \end{aligned}$$