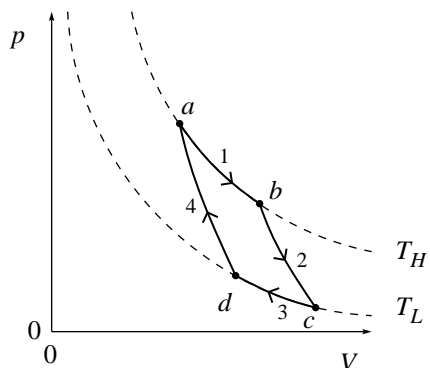# Efficiency of a Carnot engine

The Carnot cycle makes an engine. The p-V diagram below sketches the operation of a Carnot engine, where the "working fluid" that expands and contracts within the cylinder is an ideal gas.
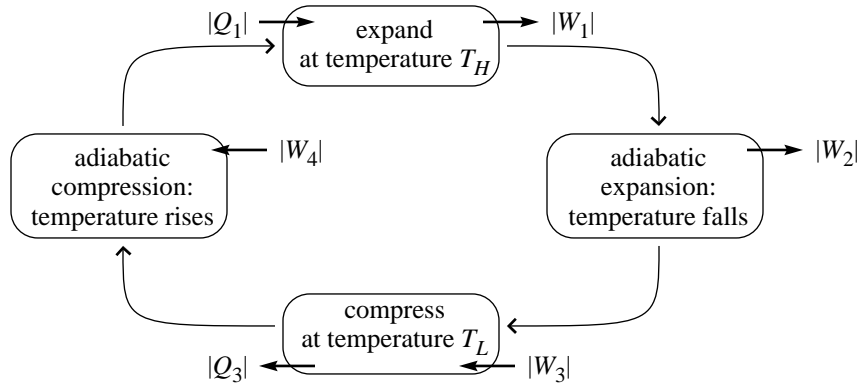

Here the high temperature T_H and the low temperature T_L are temperatures as measured on an ideal gas thermometer, that is

$$T = \frac{pV}{NR}.$$

One cycle of the Carnot engine acts as follows:

- **Leg 1:** Isothermal expansion at high temperature. As the gas expands, it lifts a big pile of sand — that is, it does work $|W_1|$. We've seen that expansion normally sends the temperature down. To keep the same temperature, the gas must absorb heat $|Q_1|$ from its surroundings.

- **Leg 2:** Adiabatic expansion. As the gas expands, it lifts a big pile of sand — that is, it does work $|W_2|$. The expansion sends the temperature down. Because it's adiabatic, no heat is absorbed or ejected.

- **Leg 3:** Isothermal compression at low temperature. As the gas contracts, it lets down a small pile of sand — that is, work $|W_3|$ is done on the gas. We've seen that compression normally sends the temperature up. To keep the same temperature, the gas must eject heat $|Q_2|$ into its surroundings.

- **Leg 4:** Adiabatic compression. As the gas contracts, it lets down a small pile of sand — that is, work $|W_4|$ is done on the gas. The compression sends the temperature up. Because it's adiabatic, no heat is absorbed or ejected.

$|Q_1|$ → expand at temperature T_H → $|W_1|$

adiabatic compression: temperature rises ← $|W_4|$

adiabatic expansion: temperature falls → $|W_2|$

compress at temperature T_L

$|Q_3|$ ← $|W_3|$

Overall, the gas absorbs some heat and ejects some heat. The gas does some work and some work is done on it. The "net work done by the gas" is the work done by the gas minus the work done on the gas. We're interested in getting as large a net work as possible for a given amount of heat absorbed. That is, we're interested in getting a large

$$\text{efficiency} = \frac{\text{net work done by gas}}{\text{heat absorbed by gas}} = \frac{|W_1| + |W_2| - |W_3| - |W_4|}{|Q_1|}. \tag{1}$$

(James Watt had a similar measure that he called "duty": the work in foot-pounds generated by a bushel of coal. I actually prefer the name "duty" to "efficiency", but we're stuck with "efficiency".)

To find the relevant quantities, remember that the work done by a quasistatic fluid is

$$\int p(V) \, dV.$$

⟦This formula applies only if the gas can be considered at equilibrium while the volume is changing. If there are currents in the gas, then one cannot even define a pressure in the way we've been using the term! Such currents will generally result in internal friction which throws off the work and heat calculations we'll do soon. If the piston experiences friction rubbing against the cylinder walls, then the force will not be the same as pressure × area. In short, this calculation will find the efficiency of an "ideal engine" using the working fluid of an "ideal gas". Real engines with friction and currents and so forth will be less efficient.⟧

Leg 1: On this isothermal expansion leg, the temperature is T_H so

$$p(V) = \frac{NRT_H}{V}. \qquad (2)$$

The work done by the gas is then

$$
\begin{aligned}
W_1 &= \int_a^b p(V)\, dV \\
&= \int_a^b \frac{NRT_H}{V}\, dV \\
&= NRT_H \int_a^b \frac{dV}{V} \\
&= NRT_H \left[\ln(V) \right]_a^b \\
&= NRT_H \ln \frac{V_b}{V_a} \qquad (3)
\end{aligned}
$$

Remember that $V_b > V_a$, so this quantity is positive, as expected. (In leg 1, the gas *does* work by lifting something.)

The heat absorbed merely counteracts the effect of the work done, so

$$|Q_1| = |W_1|. \qquad (4)$$

(This derivation relies upon the already-stated fact that the energy of a given sample of an ideal gas is a function of temperature alone, not of temperature and volume.)

Leg 2: On this adiabatic expansion leg, the pressure and volume are related through

$$pV^\gamma = \text{constant} = p_b V_b^\gamma = p_c V_c^\gamma. \qquad (5)$$

Before I calculate the work, I'm going to find a relation between temperature and volume that will turn out to be useful later:

$$
\begin{aligned}
p_b V_b^\gamma &= p_c V_c^\gamma \\
p_b V_b V_b^{\gamma-1} &= p_c V_c V_c^{\gamma-1} \\
NRT_H V_b^{\gamma-1} &= NRT_L V_c^{\gamma-1} \\
\frac{T_H}{T_L} &= \left(\frac{V_c}{V_b} \right)^{\gamma-1} \qquad (6)
\end{aligned}
$$

We've already said that an increase in volume will result in a decrease in temperature (for example, in releasing air from a bicycle tire), and this equation tells us quantitatively what that temperature change is.

With this piece of bookkeeping out of the way, we go ahead and find the work done by the gas:

$$
\begin{aligned}
W_2 &= \int_b^c p(V)\,dV \\
&= \text{constant} \int_b^c \frac{1}{V^\gamma}\,dV \\
&= \text{constant} \left[\frac{1}{-\gamma + 1} V^{(-\gamma+1)} \right]_b^c \\
&= \frac{1}{-\gamma + 1} \left[\frac{\text{constant}}{V_c^{\gamma-1}} - \frac{\text{constant}}{V_b^{\gamma-1}} \right] \\
&= \frac{1}{-\gamma + 1} \left[\frac{p_c V_c^\gamma}{V_c^{\gamma-1}} - \frac{p_b V_b^\gamma}{V_b^{\gamma-1}} \right] \\
&= \frac{1}{-\gamma + 1} \left[p_c V_c - p_b V_b \right] \\
&= \frac{1}{-\gamma + 1} NR \left[T_L - T_H \right] \\
&= \frac{NR}{\gamma - 1} \left(T_H - T_L \right).
\end{aligned}
\tag{7}
$$

Once again, and again as expected, this quantity is positive.

Leg 3: In the isothermal compression leg, the calculation of work done by the gas is performed exactly as it was in leg 1, but of course with different variables plugged in. Referring to equation (3) we find

$$
W_3 = NRT_L \ln \frac{V_d}{V_c}.
\tag{8}
$$

Because $V_d < V_c$, this quantity — the work done *by* the gas — is *negative*. The work done *on* the gas is

$$
|W_3| = -NRT_L \ln \frac{V_d}{V_c} = NRT_L \ln \frac{V_c}{V_d}.
\tag{9}
$$

Furthermore, just as in leg 1,

$$
|Q_3| = |W_3|.
\tag{10}
$$

Leg 4: In the adiabatic compression leg, the calculations are performed exactly as they were in leg 2, but of course with different variables plugged in. Referring to equations (6) and (7) we find

$$
\frac{T_L}{T_H} = \left(\frac{V_a}{V_d} \right)^{\gamma-1}
\tag{11}
$$

and

$$
W_4 = \frac{NR}{\gamma - 1} \left(T_L - T_H \right).
\tag{12}
$$

As expected, this quantity is negative and

$$
|W_4| = -W_4 = \frac{NR}{\gamma - 1} \left(T_H - T_L \right).
\tag{13}
$$

With all of the relevant quantities known we can go ahead and calculate the efficiency

$$\text{efficiency} = \frac{|W_1| + |W_2| - |W_3| - |W_4|}{|Q_1|} = 1 + \frac{|W_2| - |W_3| - |W_4|}{|Q_1|}.$$

But our calculations show that $|W_2| = |W_4|$. (In retrospect, this is obvious for an ideal gas. We didn't need to evaluate W_2 and W_4. But lots of things are obvious in retrospect. I am not ashamed.) Using our calculated values for $|W_3|$ and Q_1, we find

$$\text{efficiency} = 1 - \frac{|W_3|}{|Q_1|} = 1 - \frac{NRT_L \ln(V_c/V_d)}{NRT_H \ln(V_b/V_a)} = 1 - \frac{T_L \ln(V_c/V_d)}{T_H \ln(V_b/V_a)}.$$

We seem to have found the final answer. But there's one more simplification that can be performed. Comparing equations (6) and (11), we find

$$\frac{T_H}{T_L} = \left(\frac{V_c}{V_b}\right)^{\gamma-1} = \left(\frac{V_d}{V_a}\right)^{\gamma-1}$$

so

$$\frac{V_c}{V_b} = \frac{V_d}{V_a}$$

or

$$\frac{V_c}{V_d} = \frac{V_b}{V_a}.$$

The two natural logs in the expression for efficiency are equal!

Our final result for the efficiency of an ideal Carnot engine using an ideal gas as a working fluid is the remarkably simple

$$\text{efficiency} = 1 - \frac{T_L}{T_H}. \tag{14}$$

This efficiency is independent of the details concerning volume and depends only on the two temperatures concerned. To get a high efficiency (always a good thing) one wants the highest possible T_H and the lowest possible T_L.