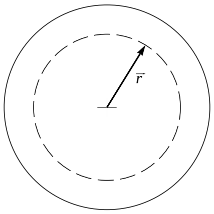
Sphere of charge with a spherical cavity

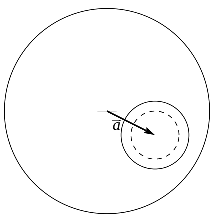
(a.)



To find $\vec{E}(\vec{r})$, use the shell theorem to realize that the charge inside the dashed line might as well be at the center, whereas the charge outside the dashed line might as well vanish. The charge inside the dashed line is $\rho\,\frac{4}{3}\pi r^3$, so

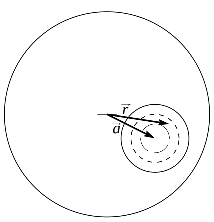
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\rho\,\frac{4}{3}\pi r^3}{r^2}\hat{r} = \frac{\rho}{3\epsilon_0} r\hat{r} = \frac{\rho}{3\epsilon_0}\vec{r}.$$

(b.)



What is the contribution to $\vec{E}(\vec{a})$ from the shell of charge represented by the dotted line? It is zero! Thus *removing* the dotted shell will have no effect on $\vec{E}(\vec{a})$. Since a sphere of charge is just the sum over many shells of charge, removing a sphere of charge centered on $\vec{a}$ also has no effect on $\vec{E}(\vec{a})$.

(c.)



Removing the dotted shell has no effect on $\vec{E}(\vec{r})$, but removing the dashed shell does!

$$\vec{E}(\vec{r}) = \frac{\rho}{3\epsilon_0}\vec{r} - \vec{E}[\text{due to the dashed shells between } \vec{r} \text{ and } \vec{a}].$$

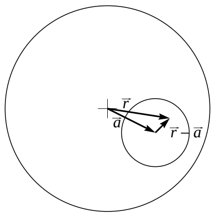
But

$$\vec{E}[\text{due to the dashed shells between } \vec{r} \text{ and } \vec{a}] = \frac{\rho}{3\epsilon_0}(\text{vector from } \vec{a} \text{ to } \vec{r}) = \frac{\rho}{3\epsilon_0}(\vec{r}-\vec{a}).$$

Hence

$$\vec{E}(\vec{r}) = \frac{\rho}{3\epsilon_0}\vec{r} - \frac{\rho}{3\epsilon_0}(\vec{r}-\vec{a}) = \frac{\rho}{3\epsilon_0}\vec{a}.$$

Alternative approach to (b.) and (c.)



A big sphere of positive charge density $+\rho$ with a little spherical hole centered on $\vec{a}$ is equivalent to a big holeless sphere of positive charge density $+\rho$ plus a little sphere of negative charge density $-\rho$. Thus the electric field at point $\vec{r}$ is the sum

$$\begin{aligned}\vec{E}(\vec{r}) &= \vec{E}[\text{due to big positive holeless sphere}] + \vec{E}[\text{due to little negative sphere}] \\ &= \frac{+\rho}{3\epsilon_0}\vec{r} + \frac{-\rho}{3\epsilon_0}(\vec{r}-\vec{a}) \\ &= \frac{\rho}{3\epsilon_0}\vec{a}.\end{aligned}$$

Grading: There are many ways to solve this problem. Grade generously and consistently.