Space and Time in Special Relativity

Contents

1 Introduction

You've studied space and time in special relativity before. This essay simply gives you a refresher and a new perspective.

The basic idea is this: Common sense — going back to Galileo — tells us that any inertial reference frame is as good as any other, and experiment (for example the classic experiments by Trouton and Noble) backs up that common sense.

But experiment also shows that the speed of light is the same in all inertial reference frames, which is certainly *not* in accord with common sense.

Now speed is defined as

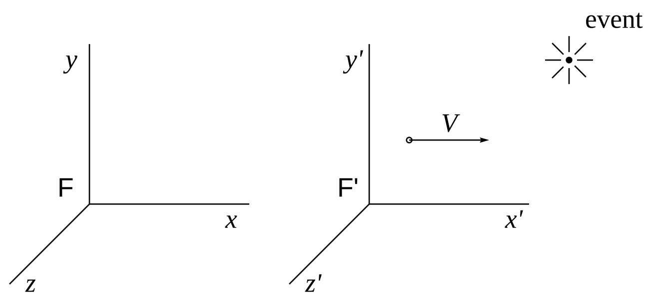
$$\text{speed} = \frac{\text{distance traveled}}{\text{time elapsed during that travel}}.$$

So if speed doesn't behave in the common-sense way then it must be because (a) time doesn't behave in the common-sense way, or (b) distance doesn't behave in the common-sense way, or (c) both.

2 Lorentz transformation

The role of derivations in physics. There are many ways to derive the Lorentz transformation from the two observations that "any inertial reference frame is as good as any other" and that "the speed of light is the same in all inertial reference frames". (I ask you to find one of them in the problem on page 5. If you like history, you can go to the Einstein Papers Project and look up his derivation from 1905.) I personally like the derivation that follows, surely in part because I invented it. You might find this derivation too subtle, or unappealing in some other way. If so, I encourage you to question but not to sweat. Derivations underscore the logical structure of the physics ideas, but it's more important for you to learn how to use the Lorentz transformation, to listen to what the Lorentz transformation is telling us about nature. (And, more concretely, I'm not going to ask you exam questions about the derivation.)

Situation. Inertial frame F' moves at constant speed V relative to inertial frame F, and the two frames coincide at time $t = t' = 0$.



An event has space-time coordinates (x, y, z, t) in frame F and space-time coordinates (x', y', z', t') in frame F'. How are the two sets of coordinates related?

Transverse coordinates. It's clear that $y' = y$ and $z' = z$. It's equally clear that the coordinates y, z will have no effect on the coordinates x, t.

Linearity. If a ball moves with constant speed in frame F ($x = A + Bt$) then (because "any inertial reference frame is as good as any other") it moves with constant speed in frame F' ($x' = C + Dt'$). From this, linear algebra books prove (it's surprisingly difficult) that for any event

$$\begin{aligned} x' &= b_{11}x + b_{12}t \\ t' &= b_{21}x + b_{22}t. \end{aligned}$$

The coefficients b_{11}, b_{12}, b_{21}, and b_{22} depend on the frame speed V, but do not depend on x or t: they are the same for all events.

I like to write this relation as

$$x' = a_{11}(\beta)x + a_{12}(\beta)ct \tag{1}$$

$$ct' = a_{21}(\beta)x + a_{22}(\beta)ct. \tag{2}$$

where $\beta = V/c$ is dimensionless and ct has the dimensions of x. (Some people like to write the product ct as x_0, but that strikes me as overkill at this point.)

Our job is to find these four functions $a_{11}(\beta)$, $a_{12}(\beta)$, $a_{21}(\beta)$, and $a_{22}(\beta)$.

Use uniform motion of frames. If $x' = 0$, our situation says that $x = Vt$ or, what is the same thing, $x = \beta ct$. But equation (1) says that if $x' = 0$, then

$$x = -\frac{a_{12}(\beta)}{a_{11}(\beta)}ct.$$

We conclude that

$$a_{12}(\beta) = -\beta a_{11}(\beta).$$

If $x = 0$, our situation says that $x' = -Vt'$ or, what is the same thing, $x' = -\beta ct'$. But dividing equation (1) by equation (2) says that if $x = 0$, then

$$\frac{x'}{ct'} = \frac{a_{12}(\beta)}{a_{22}(\beta)}.$$

We conclude that

$$a_{12}(\beta) = -\beta a_{22}(\beta),$$

which, combined with last paragraph's result, gives

$$a_{11}(\beta) = a_{22}(\beta).$$

In summary, the transformation equations must have the form

$$x' = a_{11}(\beta)(x - \beta ct) \tag{3}$$

$$ct' = a_{21}(\beta)x + a_{11}(\beta)ct. \tag{4}$$

Use invariance of light speed. The speed of light is the same in both frames. That is, if $x = ct$ then $x' = ct'$. For events on the light front, equations (1) and (2) become

$$\begin{aligned} ct' &= a_{11}(\beta)ct + a_{12}(\beta)ct \\ ct' &= a_{21}(\beta)ct + a_{22}(\beta)ct. \end{aligned}$$

The immediate consequence is that $a_{11}(\beta)+a_{12}(\beta) = a_{21}(\beta)+a_{22}(\beta)$ which, combined with previous results, gives $a_{12}(\beta) = a_{21}(\beta)$.

In summary, the transformation equations must have the form

$$x' = a_{11}(\beta)(x - \beta ct) \tag{5}$$

$$ct' = a_{11}(\beta)(-\beta x + ct). \tag{6}$$

Use inverse transformation. We've been talking about how to transform coordinates in frame F to coordinates in frame F'. But of course frame F' is just as good a frame as frame F. So to convert from coordinates in frame F' to coordinates in frame F, just use the same equation except replace β with $-\beta$:

$$x = a_{11}(-\beta)(x' + \beta ct') \tag{7}$$

$$ct = a_{11}(-\beta)(\beta x' + ct'). \tag{8}$$

Well of course we could also solve the two equations (5) and (6) algebraically for x and ct. The algebra is straightforward and results in

$$\begin{aligned} x &= \frac{x' + \beta ct'}{a_{11}(\beta)(1-\beta^2)} \\ ct &= \frac{\beta x' + ct'}{a_{11}(\beta)(1-\beta^2)}. \end{aligned}$$

Comparing these two equation pairs demonstrates that

$$a_{11}(\beta)a_{11}(-\beta) = \frac{1}{1-\beta^2}. \tag{9}$$

Use isotropy of space. Here are two questions:

(1) A clock moves at speed V to the east past a standing person. When the clock passes the person, it reads 0 seconds. What time has passed, in the person's frame, when the clock strikes 14 seconds?

(2) A clock moves at speed V to the west past a standing person. When the clock passes the person, it reads 0 seconds. What time has passed, in the person's frame, when the clock strikes 14 seconds?

These two questions differ only by the substitution of "west" for "east", and since we don't expect the direction of the clock's motion to make any difference, we expect these two questions to have the same answer. You might call this "east/west symmetry", or "right/left symmetry", or "reflection symmetry", or even by the high-falutin' name of "isotropy of space", but what you call it matters not at all. What matters is that the two different experiments are described differently:

(1) The person stands at the origin of frame F, the clock rests at the origin of frame F'. The event "clock strikes 14 seconds" has coordinates $x' = 0$, $t' = 14$ s, so (by equation 8), the answer to the question is $t = a_{11}(-\beta)(14\text{ s})$.

(2) The person stands at the origin of frame F', the clock rests at the origin of frame F. The event "clock strikes 14 seconds" has coordinates $x = 0$, $t = 14$ s, so (by equation 6), the answer to the question is $t' = a_{11}(\beta)(14\text{ s})$.

East/west symmetry demands that these two questions have the same answer, so $a_{11}(\beta) = a_{11}(-\beta)$ and equation (9) results in

$$a_{11}(\beta) = \frac{1}{\sqrt{1-\beta^2}}. \tag{10}$$

To conclude, this argument has determined that

$$x' = \frac{x - \beta ct}{\sqrt{1-\beta^2}} \tag{11}$$

$$ct' = \frac{-\beta x + ct}{\sqrt{1-\beta^2}}. \tag{12}$$

This pair is called the "Lorentz transformation". (Why this centerpiece of Einstein's relativity theory is named after Dutch physicist Hendrik Lorentz is a question of history not to be pursued here.)

We will usually write these as

$$x' = \frac{x - Vt}{\sqrt{1-(V/c)^2}}$$

$$t' = \frac{t - Vx/c^2}{\sqrt{1-(V/c)^2}}.$$

If there are two events, then they are separated by

$$\Delta x' = \frac{\Delta x - V\Delta t}{\sqrt{1-(V/c)^2}}$$

$$\Delta t' = \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}}.$$

Problem 1: We traced through the logic by first demanding the invariance of light speed, then demanding that inverse transformation works correctly. Work through the derivation again but this time invert those two demands.

Acknowledgments: Rob Owen made suggestions concerning the "isotropy of space" step, and my students in the fall semester 2016 offering of Physics 212, Modern Physics, refined the arguments of that same step.

3 Speed transformation

Well, if space and time transform between frames through the non-common-sensical Lorentz transformation, and if speed is just the quotient of a space with a time, then how do speeds transform?

A bird flies with uniform speed along the x-axis. Two events happen along the bird's trajectory: perhaps (1) it raises its wings and (2) it lowers its wings. In frame F these two events are separated by distance Δx and time Δt; in frame F' they are separated by distance $\Delta x'$ and time $\Delta t'$. The speed of the bird is v_b in frame F and v'_b in frame F', where

$$v_b = \frac{\Delta x}{\Delta t}$$

and

$$\begin{aligned} v'_b &= \frac{\Delta x'}{\Delta t'} && \text{(use Lorentz transformation to find...)} \\ &= \frac{\Delta x - V\Delta t}{\Delta t - V\Delta x/c^2} && \text{(divide numerator and denominator by } \Delta t \text{ to find...)} \\ &= \frac{v_b - V}{1 - v_b V/c^2} \end{aligned}$$

Examples:

		common sense	correct relativistic formula
$v_b = 100$ mph	$V = 20$ mph	$v'_b = 80$ mph	$v'_b = 80.000\,000\,000\,000\,02$ mph
$v_b = c$	$V \neq c$	$v'_b = c - V$	$v'_b = c$
$v_b = -\frac{3}{4}c$	$V = \frac{3}{4}c$	$v'_b = -\frac{3}{2}c$	$v'_b = -\frac{24}{25}c$

Problem 2: Apple on Train. A train moves east at speed v_t relative to the earth. Someone within the train tosses an apple east at speed v_a relative to the train. In the earth's frame the apple must move at some speed *faster* than the train. Call this "excess speed" by the name v_a^x.

a. Show that
$$v_a^x = v_a \frac{1 - v_t^2/c^2}{1 + v_a v_t/c^2}.$$

b. This formula shows that v_a^x is always *less* than v_a. Justify this fact qualitatively using the concepts of time dilation and length contraction.

c. If $v_t \to c$, what does v_a^x approach?

4 Time dilation, length contraction, and the relativity of synchronization

Start with Lorentz transformation

$$x' = \frac{x - Vt}{\sqrt{1-(V/c)^2}} \tag{13}$$

$$t' = \frac{t - Vx/c^2}{\sqrt{1-(V/c)^2}}. \tag{14}$$

If there are two events, then they are separated by

$$\Delta x' = \frac{\Delta x - V\Delta t}{\sqrt{1-(V/c)^2}} \tag{15}$$

$$\Delta t' = \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}}. \tag{16}$$

Further, frame F' is just as good an inertial frame as frame F, so we can transform from coordinates in F' to coordinates in F simply by replacing every V with a $-V$:

$$\Delta x = \frac{\Delta x' + V\Delta t'}{\sqrt{1-(V/c)^2}} \tag{17}$$

$$\Delta t = \frac{\Delta t' + V\Delta x'/c^2}{\sqrt{1-(V/c)^2}}. \tag{18}$$

To derive **time dilation**, think about what time dilation means: The single moving clock ticks twice — two events. The clock is stationary in frame F', so these two ticks are separated by $\Delta x' = 0$ and $\Delta t' = T_0$. In frame F, the time elapsed T is given by equation (18), so

$$T = \frac{T_0}{\sqrt{1-(V/c)^2}}.$$

To derive **length contraction**, think about measuring the length of a rod moving in the laboratory: arrange for two events, simultaneous in the laboratory frame ($\Delta t = 0$), to occur at the two ends of the moving rod. These events are separated by length $L = \Delta x$ in the laboratory frame. In the rod's frame F', the two events are not simultaneous, but they don't need to be: the distance between them is $\Delta x' = L_0$, regardless of $\Delta t'$, because the rod is at rest. Equation (15) gives the relationship

$$L_0 = \frac{L}{\sqrt{1-(V/c)^2}}$$

which is usually written

$$L = L_0\sqrt{1-(V/c)^2}\,.$$

The third principle is **relativity of synchronization**. A pair of moving clocks ticks simultaneously ($\Delta t' = 0$) in their own frame (F'), and the distance between them in that frame is $L_0 = \Delta x'$. In the laboratory frame the ticks are not simultaneous: according to equation (18), those two ticks are separated by a time

$$\frac{VL_0/c^2}{\sqrt{1-(V/c)^2}}.$$

But just because those two ticks are separated by this much time doesn't mean that this is the difference in the times announced by these two clocks: The two clocks are ticking slowly (time dilation) so the difference in time announced is the smaller time $\sqrt{1-(V/c)^2}$ times the above, namely

$$\frac{L_0 V}{c^2}.$$

For a pair of moving clocks (synchronized in their own frame) the *rear* clock is set *ahead*.

5 Pole in the barn

Most barns have two doors, so that you can pull a trailer into the barn, stop and unload it, and then drive out again without backing up. The farm where I grew up in Pennsylvania had a barn extending exactly 100 feet between its two doors.

One day a world champion pole vaulter came to visit our farm. He carried his favorite pole which, by coincidence, was also exactly 100 feet long. The champion boasted that he was so fast that, even carrying his pole horizontally, he could run right through our barn at the speed of $V = \frac{4}{5}c$.

"At that speed," he assured my father, "my pole will be length contracted until it's only

$$\sqrt{1-(V/c)^2}\, L_0 = \tfrac{3}{5}(100 \text{ ft}) = 60 \text{ ft}$$

long. I'll be able to fit it completely within your old barn! Look, if you don't believe it, put me to the test. Station one of your sons at the front door and the other one at the rear door. Start with both doors closed, and open each one for just long enough to allow me through. You'll see. There will be a time when both doors are shut and my pole is completely enclosed within your barn."

My father was no dummy. He rubbed his chin and looked puzzled and thoughtful for a minute. "Okay, you're on," he told the pole vaulter. "I'll station my boys. But there's just one thing I don't understand: Sure, in the barn's frame your pole will be length contracted. But in *your* frame the *barn* is moving. In *your* frame my barn is 60 feet wide and your pole is 100 feet long. How are you going to fit that long pole of yours into my stubby little barn?"

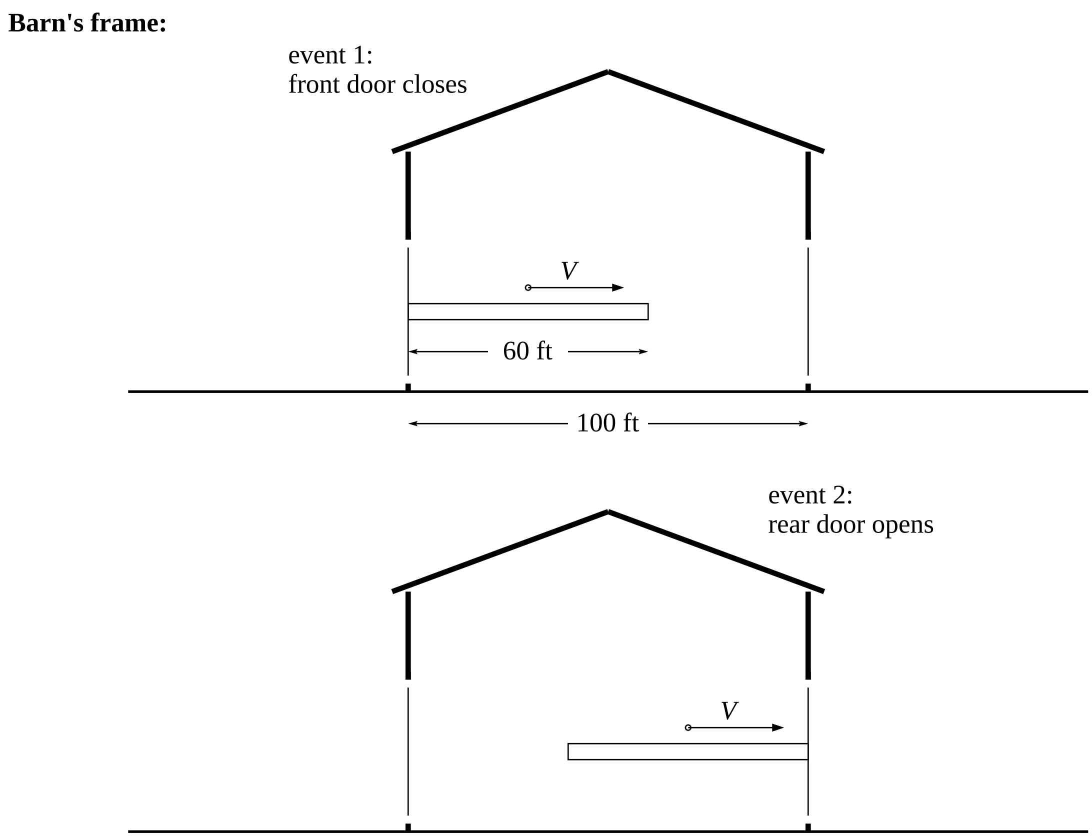
Now it was the champion's turn to be puzzled. In fact, he looked frightened and just a little greenish. I could see the sweat bead up on his forehead, and he lost his confident swagger. He wanted to bail out. My older brother went up and whispered a few words into the vaulter's ear. They huddled in quiet conversation for a few minutes, and then the vaulter regained most of his lost confidence. He carried out the feat flawlessly.

What did my brother tell the vaulter?

"You have to think about time and simultaneity also," my brother whispered, "not just about length."

"If I'm going to risk my life," the vaulter responded, "I'll need a lot more detail than that."

"Fair enough," said my brother, as he drew a figure onto the barnyard dust with a stick, "Here's a sketch in the barn's frame. The two important events are when the front door closes and when the rear door opens. Between those two events you're completely enclosed within the barn."



Now, what are Δx and Δt between these two events?"

"That's easy," replied the vaulter. "Δx is just the length of the barn, 100 feet, and Δt is just the time I need to run 40 feet. Since I'll be running at $V = \frac{4}{5}c$, the time required is

$$\Delta t = \frac{40\text{ ft}}{\frac{4}{5}c} = (50\text{ ft})/c.$$

Now I just need to find the speed of light in feet per second. Let me Google it ..."

"No, wait," interrupted my brother. "Before performing the arithmetic, let's work with the symbol c. Maybe we won't need to actually figure out the value in seconds."

"Well, if you say so," replied the vaulter dubiously. "I guess next you'll want to know the separations $\Delta x'$ and $\Delta t'$ in my frame."

"Let's begin by finding only $\Delta t'$. I think that's all we'll need. How can we find it?"

“We use the Lorentz transformation, of course.” The vaulter was now looking more peeved than anxious. He took my brother’s stick and began tracing equations into the dust.

$$\begin{aligned}
\Delta t' &= \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}} \\
&= \frac{(50\text{ ft})/c - \frac{4}{5}(100\text{ ft})/c}{\frac{3}{5}} \\
&= \frac{(50\text{ ft})/c - (80\text{ ft})/c}{\frac{3}{5}} \\
&= \frac{-(30\text{ ft})/c}{\frac{3}{5}} \\
&= -(50\text{ ft})/c.
\end{aligned}$$

“But that’s . . . what . . . a negative number. What could that mean?”

“It means,” said my brother, “that in the barn’s frame, where Δt is positive, first the front door closes, and then the rear door opens. Between those two events, the pole is completely enclosed within the barn. Whereas in the vaulter’s frame . . . .”

“In my frame,” the vaulter jumped in, “those two events happen in the opposite sequence: *first* the rear door opens, and *second* the front door closes. Can that be true? Can it really be that the pole is never enclosed within the barn in my frame?”

“Sure it can be true,” assured my brother. “As my kid brother likes to say, ‘Two events that occur in one sequence in one reference frame might occur in the opposite sequence in a different reference frame.’ ”

Problem 3: The pair of sketches in the barn’s frame on page 10 shows (1) the pole’s butt entering the front of the barn and (2) the pole’s tip leaving the rear of the barn. Draw a similar pair of sketches showing these two events in the vaulter’s frame. Describe in words the differences between the situation in the barn’s frame and in the vaulter’s frame.

Problem 4: Repeat the calculation of this section with a barn and a pole of rest length L_0 (instead of 100 feet), and a pole vaulter of speed V (instead of $\frac{4}{5}c$).

6 The tossed tomato: Causality and speed limits

We have seen that, given two events, it’s possible that in some frames event #1 comes first, in other frames event #2 comes first, and in one frame the two events are simultaneous.

But suppose that event #1 causes event #2. (For example: event #1 is “I toss a tomato,” event #2 is “tomato splatters over the wall.”) In this case you’d certainly think that event #1 has to occur before event #2 in all frames! Let's make this assumption and see where it takes us.

Define

$$\frac{\Delta x}{\Delta t} = \text{ speed of the causal signal } = v_s.$$

(In our example, the "causal signal" is the tomato.)

Our assumption is $\Delta t' > 0$ in all reference frames, so, by the Lorentz transformation,

$$\Delta t' = \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}} > 0 \quad \text{for all frames.}$$

Thus

$$\begin{aligned} \Delta t &> V\Delta x/c^2 \\ 1 &> Vv_s/c^2 \\ c^2 &> Vv_s \end{aligned}$$

If the causality assumption is true, then this holds for all frames, and for all causal signals.

Suppose the causal signal is light, so that $v_s = c$. Then

$$c > V,$$

that is, all reference frames travel at less than the speed of light.

Pick one such reference frame moving just slower than c: that is $V = c - \epsilon$, where ϵ can be as small as you wish as long as it's greater than zero. Then for any causal signal

$$\begin{aligned} c^2 &> Vv_s \\ c^2 &> (c-\epsilon)v_s \\ c &> \left(1 - \frac{\epsilon}{c}\right) v_s \end{aligned}$$

Let $\epsilon \to 0$ to find

$$c \geq v_s,$$

that is, any causal signal travels at less than or equal to the speed of light.

Has this prediction been verified? Well, sending information is not just fun to speculate about, it's a multi-billion dollar industry. The telecommunications industry, the television industry, the Internet industry, the computer industry, the courier industry (and, from the past, the Pony Express): all, ultimately, are about sending information from place to place as quickly as possible. These industries work tirelessly to find ways of sending information quickly, but none has figured out a way of sending it faster than the speed of light.

Anything else? At the CERN laboratory near Geneva, Switzerland, scientists have figured out how to push a single electron so hard and so many times that, if common-sense notions of space and time were correct, the electron would be traveling 904 times the speed of light. But in fact no electron at CERN has traveled as fast as or faster than light. (The maximum speed achieved so far is $0.999\,999\,999\,997\,c$.)

So despite the enormous monetary and scientific rewards that would be showered upon anyone sending a causal signal faster than light, no one has ever been able to do so: good evidence in support of both relativity and our causality assumption.

7 Rigidity, straightness, and strength

Here is a proposal for sending a signal faster than the speed of light. . . in fact, for sending it instantaneously: Push the left end of a rod. . . the right end moves at the same time! Well, not quite. When you push the left end of the rod, you move the first atom in a long chain of atoms that makes up the rod. A short time later, the first atom pushes the second, then the second pushes the third, and so forth. This push moves down the rod and reaches the end at a speed that is very fast by human standards (for a steel rod it moves at about 3 miles/second), so we don't notice it. But the speed is very slow compared to the speed of light. *There is no such thing as a perfectly rigid rod.*

Ivan mounts a straight rod horizontally on three pegs, and he fits the base of each peg with a firecracker that can cause it to crumble into bits. He arranges for the three pegs to crumble simultaneously, and at that same instant the rod begins to fall down. The rod is always straight and always horizontal. Veronica moves to the right relative to Ivan. In Veronica's reference frame these three events are not simultaneous: First the right peg crumbles and the right end of the rod begins to fall, second the middle peg crumbles and the middle of the rod begins to fall, third the left peg crumbles and the left end of the rod begins to fall. Between the first and second events the rod must be curved in Veronica's reference frame. *A rod that is straight in one reference frame may be curved in another.*

A bicycle wheel is set up on a rack and spun very quickly. The pieces of the wheel rim move parallel to their lengths, so they are length contracted. But the spokes move perpendicular to their lengths, so they are not length contracted. How can wheel hold together with a contracted circumference and a non-contracted radius? The answer is that it can't. When a wheel rotates fast it breaks apart and flies into pieces. A wheel made of a weak material like wood will break apart at rather slow speeds; a wheel made of a strong material like steel will break apart at higher speeds; a wheel made of a very strong material like diamond will break up at still higher speeds; but a wheel made up of *any* material will break apart a speeds much lower than those where the relativistic effect becomes noticeable. *There is no infinitely strong material.*

The relativistic limits on the rigidity and strength of materials are can be worked out quantitatively, and they are extreme. All known materials are much less rigid and much less strong than the limits allow.

8 Measuring the length of a moving rod

Veronica speeds to the right past Ivan: he says her rods are short. But in her reference frame, Ivan speeds to the left past Veronica: she says his rods are short. Isn't this a logical contradiction? No. To see why, we have to think about what's involved in measuring the length of a moving rod.

It is no big deal to measure the length of a stationary rod: simply find the position of one end, and find the position of the other, and subtract those positions to find the length:

```
        o[XXXXXXXXXXXXXXXX]o
         |<--- length --->|
```

If you wanted, you could find the position of the right end, then go out and eat lunch, then come back and find the position of the left end.

NOT SO if you wanted to find the length of a moving rod! If you find the position of the front end, then wait a while, then find the position of the rear end, the difference will be a lot shorter than the length of the rod:

```
first:     [XXXXXXXXXXXXXXXX]o ---> V
later:         o[XXXXXXXXXXXXXXXX] ---> V
                |<- short ->|
```

If you want to find the length of a moving rod, you have to find the positions of the two ends simulteneously.

And there's that troublesome word! You know that if the two position measurements are simultaneous in one frame, they will not be simultaneous in another frame.

To make this concrete, I'll measure the length of the rod by lining up eight people, named A, B, C, D, E, F, G, and H. They stand one foot apart, and each person carries a watch. They are instructed to raise their hand (and keep them up) if either end of the rod is directly in front of them when their watch strikes noon.

Process in measurers's frame:

```
            !       !
A   B   C   D   E   F   G   H
            [XXXXXXX]  ---> V
```

D and F raise their hands simultaneously, so a single sketch illustrates the situation.

Same process in rod's frame:
The rear hand raising (F) happens first

```
                                  !
                V <---  A B C D E F G H
                 [XXXXXXXXXXXXXXXX]
```

and the front hand raising (D) happens later

```
                 !   !
   V <---  A B C D E F G H
                 [XXXXXXXXXXXXXXXX]
```

F raises hand before D, and the measurers move between those two hand raisings, so two sketches are required.

The measurers say the moving rod is short.

The rod says that the measurers are close together, but they don't raise their hands simultaneously, so of course they measure a different (shorter) length.

Both statements are correct.

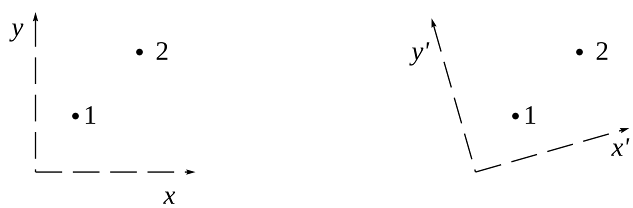
9 Interval

In the problems I have asked you to prove that for any two reference frames, regardless of their relative velocity V,

$$(c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2.$$

This quantity is called "interval" or, by those who like to sound sexy, "spacetime interval". The problem solution is nothing but algebra: I did it by starting with the right-hand side, plugging in the Lorentz transformation equations, and chugging away until I got to the left-hand side. The dramatic moment in the proof came at the final step: every intermediate step involved the relative velocity V, but in the final step the expressions involving V canceled out numerator and denominator, resulting in the left-hand expression independent of V. Here I ask, not about the mechanics of the algebraic chugging, nor about the drama of the final step, but about the physical significance of this result.

Here's an analogy involving not relativistic spacetime, but ordinary three-dimensional space. In fact, to make it easier to draw, I'll use two-dimensional space. These diagrams show two points in space.



The two diagrams show the same two points, but each shows a different choice of coordinate axes: On the left, the two points are separated by Δx and Δy. On the right, the two points are separated by different coordinates $\Delta x'$ and $\Delta y'$. These are the same two points, with the same separation; the different coordinates reflect the different coordinate axes, not any difference in the physical points.

Whether we call a tree by the English name "tree" or by the German name "baum" doesn't make any difference to the tree; it's the same tree regardless of name used. The name merely reflects the human convention of language used, nothing intrinsic to the tree itself. Similarly, whether we call the separation between two points by the coordinates Δx, Δy or by the coordinates $\Delta x'$, $\Delta y'$ doesn't make any difference to the separation; it's the same separation regardless of coordinates used. The coordinates merely reflect the human choice of axes used, nothing intrinsic to the separation itself. This is not to belittle either language or coordinates: both are essential to human communication. It *is* to point out that their role is neither more nor less than human communication.

If you enjoy trigonometry, you will enjoy showing that, if coordinate axis x' is rotated by angle θ relative to coordinate axis x, then the coordinates are related through

$$\begin{aligned}\Delta x' &= +\cos\theta\,\Delta x + \sin\theta\,\Delta y\\ \Delta y' &= -\sin\theta\,\Delta x + \cos\theta\,\Delta y.\end{aligned}$$

Starting from this, you could prove that

$$(\Delta x)^2 + (\Delta y)^2 = (\Delta x')^2 + (\Delta y')^2.$$

Your proof would have the same character as the proof that interval is the same in both frames: There's an algebraic chug, each step of which involves the angle θ, and then a dramatic surprise at the last step when several θs cancel out and you uncover the above θ-independent result.

A quantity that is the same regardless of reference frame is called an "invariant" — it's a quantity that doesn't vary from from frame to the next. Invariants are significant because we believe (for good reason) that physical effects depend only upon the physical situation, and not upon the human choice of coordinate system. For example, if the two points of the figure were occupied by spheres of mass m_1 and m_2, you know that the gravitational force between them would have magnitude

$$G\frac{m_1 m_2}{\Delta x^2 + \Delta y^2},$$

which is invariant. If I suggested to you that the gravitational force between the two points instead had magnitude

$$G\frac{m_1 m_2}{\Delta x^2 + 7\Delta y^2},$$

then you would laugh in my face. If this formula were true, then the gravitational force would depend upon the human choice of coordinate system — which is just as absurd as suggesting that the force depends upon whether it is described using the English word "gravity" or the German word "schwerkraft". The time-dependent interaction between two events in relativistic mechanics is more intricate than the static interaction between two points in space, but the same idea applies: the interaction has to depend upon the invariant interval, not upon any frame-dependent quantity.

In the case of separation of two points in space the significance of the invariant

$$(\Delta x)^2 + (\Delta y)^2$$

is perfectly clear just from appearance: it's the square of the distance read off from a ruler stretched between the two points. In the case of separation of two events in spacetime the significance of the invariant interval is not clear from visual inspection — at least not to me. (The famous book *Gravitation* by Charles Misner, Kip Thorne, and John A. Wheeler attempts to make this straightforward on pages 11 and 58, but I have always found these pages to be opaque.) But it plays the same role in deciding which forms of interaction are physically permissible.

Distance in space is easy to pictorialize and interpret — it's just common sense.

Interval in spacetime is difficult to pictorialize and interpret — but you shouldn't expect it to be, because nothing in relativity is common sense.

Yet it is our task as scientists to develop such pictures and interpretations. The following four problems begin this development.

Problem 5. A burst of light is created in a flashlight, travels in a straight line, and is absorbed at a wall on the opposite side of the room 10 meters away. What is the interval between creation and absorption?

Problem 6. A burst of light is created in a supernova, travels in a straight line, and is absorbed at a planet on the opposite side of the galaxy 100,000 light-years away. What is the interval between creation and absorption?

Problem 7. A train departs from New York and arrives in Chicago, 713 miles away, 19 hours later. Is the interval between departure and arrival positive, zero, or negative? (Such a separation — in which the "space parts" $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ are smaller than the "time parts" $(c\Delta t)^2$ — is called "timelike". If two events are separated by a timelike interval, there is some reference frame [in this example the train's reference frame] in which those two events happen at the same location. The first event will precede the second in all reference frames, and the first event might or might not cause the second.)

Problem 8. In the "pole in the barn" paradox, call "front door closes" event number 1 and "rear door opens" event number 2. Is the interval between these two events positive, zero, or negative? (Such a separation — in which the "space parts" $(\Delta x)^2+(\Delta y)^2+(\Delta z)^2$ are larger than the "time parts" $(c\Delta t)^2$ — is called "spacelike". If two events are separated by a spacelike interval, there is some reference frame in which those two events happen at the same time. The sequence of the two events might differ in different reference frames, and one event cannot cause the other.)

Acknowledgment: This section grew out of questions from Rainie Heck, Oberlin College class of 2020.

10 The twin paradox

Sirius is the brightest star in the night sky, a brilliant blue-white star located just below the constellation Orion. Any astronomy textbook will tell you that Sirius is located eight light-years away from Earth. This means that (in the Earth's frame) it takes eight years for light, traveling at 186 000 miles/second, to fly from Sirius to Earth. We won't write down this distance in miles or in meters. Instead, we'll say instead that the distance to Sirius is (8 yr)c.

Veronica decides to take a trip from Earth to Sirius and back again, traveling at $V = \frac{4}{5}c$. Veronica's friend Ivan chooses to remain at home on Earth.

Let's get two issues out of the way at the very start: First, the Earth and Sirius move relative to each other (if nothing else, due to the Earth's orbit around the Sun) but this relative motion is so slow relative to $\frac{4}{5}c$ that we can safely ignore it, and consider Sirius to be at rest in the Earth's frame. Second, this relative motion means that the distance from Earth to Sirius changes with time, but these distance changes are so much smaller than 8 light-years that we can safely ignore them, too.

The high points of Veronica's journey, as observed from the Earth's frame, are shown on page 21. (The rectangular clocks display time measured in years.) You should be able to calculate all of these clock readings yourself: the only principles employed are the definition speed = distance/time and time dilation.

One more issue requires attention here. Veronica doesn't just step into her space ship and then instantly move at $\frac{4}{5}c$ any more than you step into your car and then instantly move at 60 miles/hour. It takes some time (a "period of acceleration") for Veronica to get up to her cruising speed. Exactly how much time it takes will depend on the kind of space ship Veronica uses, but let's say it's a week: it can't be instant, but the acceleration period can be small compared to the many years of total travel time. Similarly for the turnaround at Sirius: let's say it takes two weeks for her to slow down, turn around, and then get up to cruising speed for the return leg of journey. But since the clock readings shown in the figure are accurate only to the nearest tenth of a year anyway, these acceleration periods can safely be ignored.

The upshot is that at the end of the trip Ivan's clock has ticked off 20 years while Veronica's clock has ticked off 12 years. This applies not only to wristwatches, but also to biological clocks. Ivan will have aged 20 years and Veronica will have aged 12 years, so there will be more wrinkles on his face than on her face. Ivan explains this by saying that Veronica's moving clock ticks slowly. How does Veronica explain it?

We examine the outbound leg of the journey in Veronica's frame. As usual, changing from the Earth's frame to Veronica's frame involves four differences:

1. Instead of Veronica moving right, the Earth and Sirius move left.
2. The Earth and Sirius are closer (length contraction).
3. Clocks on the Earth and Sirius are not synchronized (relativity of synchronization).
4. The Earth and Sirius clocks tick slowly (time dilation).

I'm not going to detail how I arrived at the numbers on line A at page 22. You should do this for yourself. Be sure to notice the qualitative character of these results: The distance from Earth to Sirius is *shorter* in Veronica's frame. The Sirius clock is to the rear, so it is set *ahead*.

How much time elapses before Sirius comes to meet Veronica? This is just

$$\text{time elapsed} = \frac{\text{distance traveled}}{\text{speed}} = \frac{(4.8\text{ yr})c}{\frac{4}{5}c} = 6\text{ yr},$$

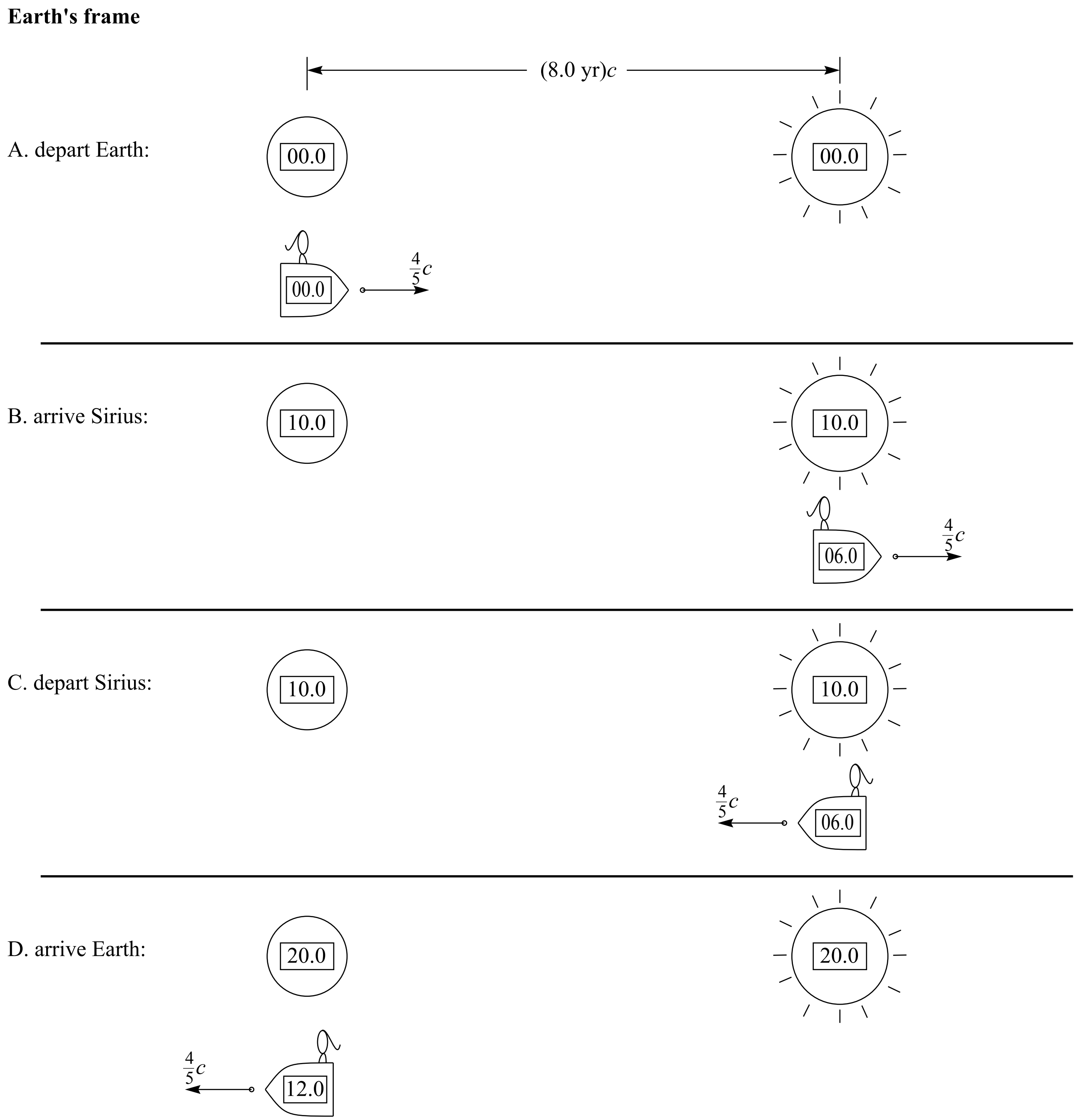
so six years elapse and Veronica's clock ticks off six years. The Earth and Sirius clocks tick slowly, of course, so they tick off only $\frac{3}{5}(6\text{ yr}) = 3.6$ yr. These results are reflected in line B at page 22.

The next step, as described from the Earth's frame, is that Veronica slows down, turns around, and heads back to Earth. She's slipping out of one inertial frame, and into another. But from Veronica's point of view, the difference is that the Earth and Sirius are no longer moving left...instead they're moving right. On the outbound leg of the journey Earth is to the *front* of Sirius, so its clock is set *behind* by 6.4 years. On the return leg Earth is to the *rear* of Sirius, so its clock is set *ahead* by 6.4 years. Compare closely lines B and C at page 22. During the turnaround, which requires two weeks for Veronica, the Earth clock jumps ahead by 12.8 years! This is the relativity of synchronization in action.

The return leg of the journey is like the outbound leg: Veronica's clock ticks off 6 years and the Earth and Sirius clocks tick off 3.6 years. Veronica finds that when she returns home, her clocks have ticked off 12 years while Ivan's have ticked off 20 years.

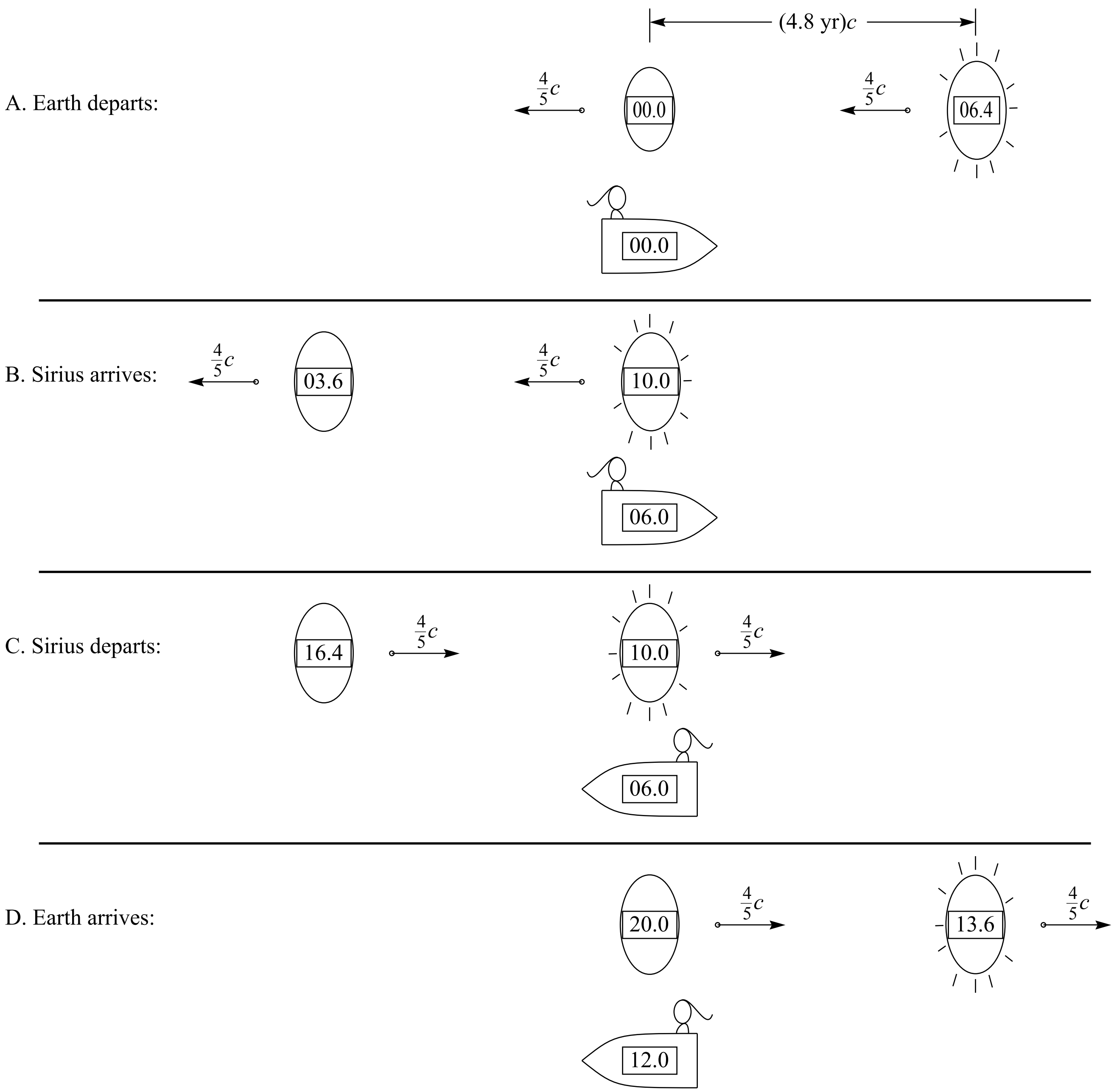
Ivan explains this difference by saying that Veronica's clock ticks slowly. Veronica explains it by saying that Ivan's clock ticks slowly, except that during the turnaround (when it changed from front clock to rear clock) it ticked very rapidly indeed.

This effect is called the "twin paradox" because if two twins go separate ways — one staying at home and the other traveling at high speed — then when they come back together the traveling twin will be younger.



Veronica's voyage to Sirius and back, from the Earth's reference frame.

Veronica's frames



Sirius's voyage to Veronica and back, from Veronica's two reference frames.

Problem 9: Keeping in touch. Before Veronica departs for Sirius at $V = \frac{4}{5}c$, she and Ivan agree to keep in touch during their long separation.

a. Ivan sends Veronica a radio message once a day. (Radio messages travel at the speed of light.) Because these messages have to catch up to Veronica, she does not receive them once a day. (This is not a relativistic effect…it happens in the common sense world as well.) Show that the message Veronica receives as she turns around at Sirius was sent by Ivan two years after Veronica left Earth. Therefore Veronica receives 1/10th of Ivan's messages on the outbound leg of her journey and 9/10th of them on the return leg.

b. Argue that Veronica receives these messages at regular intervals on the outbound leg and also at regular intervals on the return leg, but that the time between receptions is nine times longer on the outbound leg than on the return leg.

c. Veronica sends Ivan a radio message once a day. Show that the message Veronica sends from Sirius reaches Ivan when his clock reads 18 years. Show also that during Ivan's first 18 years of separation, he receives — at regular intervals — the messages sent by Veronica on the outbound leg of her journey, and that during Ivan's final 2 years of separation, he receives — again at regular intervals — the messages sent by Veronica on the return leg of her journey. What is the relation between the interval of the first 18 years and the interval of the final 2 years?

d. During their separation Ivan trains his telescope on Veronica once a day. Show that during Ivan's first 18 years he sees the image in his telescope age (at a uniform rate) by 6 years, and that in Ivan's final 2 years he sees the image age (at a uniform rate) by 6 years again.

e. During their separation Veronica trains her telescope on Ivan once a day. Show that during the outbound leg she sees the image in her telescope age (at a uniform rate) by 2 years, and that during the return leg she sees the image age (at a uniform rate) by 18 years. Her telescope image does not show a sudden jump in Ivan's age.

11 Epilogue

This is the end of the essay, but not the end of relativity.

I have spent four decades pondering and calculating and experimenting with space and time, deepening and broadening my understanding of relativity, yet still there are facets I find puzzling. For the first decade or so I found this disorienting and depressing, but I've grown to appreciate it: It would be sad indeed if I understood relativity so thoroughly that it would never again surprise or delight me.

If you're as lucky as I am, then you too will be surprised and delighted for the rest of your life.