Relativistic Limits on Rigidity and Strength

Dan Styer; © 16 September 2015

The rigidity and strength of materials is an important study in materials science. This document begins non-relativistically by considering how to measure such properties, then goes on to estimate relativistic limits.

Measuring rigidity and strength

Get a bar of material — say wood or steel — of length L and uniform cross-sectional area A. Apply a force F in an attempt to stretch the material. (In engineering parlance, this is a "tensile" force, as opposed to a "compressive" force.) The material will stretch by a length ΔL.

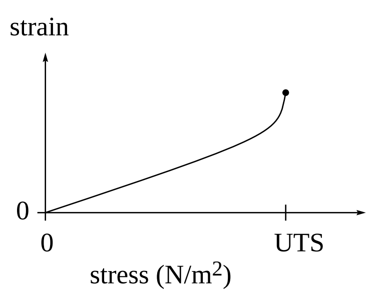
The measure of the force is called "stress":

$$\frac{F}{A} \equiv \text{stress},$$

the measure of the stretch is called "strain":

$$\frac{\Delta L}{L} \equiv \text{strain}.$$

How does the amount of stretch depend upon force applied? For many materials the result is like this:

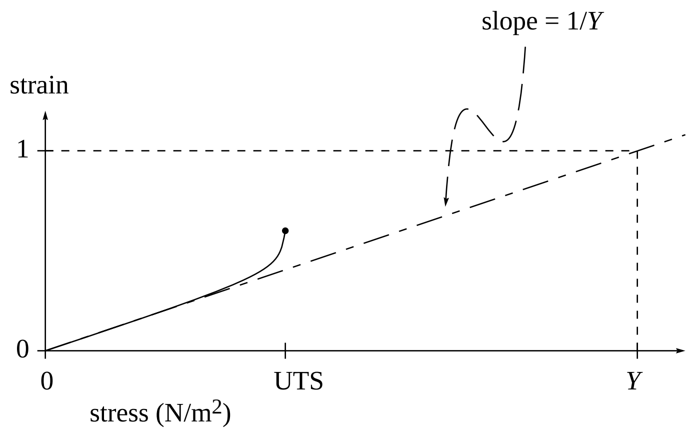


With more force (more stress) there is more stretching (more strain). For small forces this relationship is linear. For larger forces the curve deviates from linearity, and then the bar snaps in two. The stress that results in snapping is called the "ultimate tensile strength" (UTS). That is, the UTS measures a material's "strength" or "toughness".

For the linear part of the curve, the slope is given the name of $1/Y$, where Y is "Young's modulus".[1] The Young's modulus measures a material's "stiffness" or "rigidity" — if a material were perfectly rigid, it would not stretch for any force, so Y would be infinite.

[1]Thomas Young, 1773–1829, performed the first double slit interference experiments to test the wave theory of light and to measure its wavelength. He used the Rosetta Stone to decipher Egyptian hieroglyphs. The term modulus derives from the Latin word for "measure". So "Young's modulus" means "Young's measure of stiffness".

The graph below demonstrates that Y is the stress that would need to be applied to make the bar double in length — if it didn't snap (or become non-linear) first.



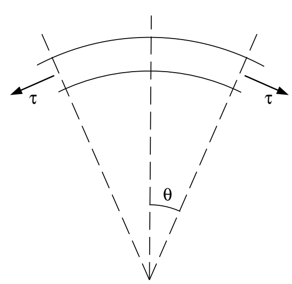
Almost all materials *do* snap before they stretch to double their initial length (silly putty and rubber bands are exceptions), so for almost all materials the UTS is less than the Young's modulus.

Here are a few experimental values, in units of MPa (that is, $\times 10^6$ N/m^2):

material	UTS	Y
silicone rubber	16 MPa	40 MPa
wood	40 MPa	11 000 MPa
steel (ASTM A36)	400 MPa	200 000 MPa
diamond	2 800 MPa	1 210 000 MPa

Circular motion (non-relativistic analysis)

A bar of material with mass per length μ is formed into a hoop of radius R, which spins with velocity v. The hoop doesn't fly apart into pieces because a tension force τ within the hoop holds it together. This free-body diagram demonstrates that the net tension force on a hoop segment subtending angle 2θ is $2\tau \sin\theta$ or, for short segments, $2\tau\theta$.



The mass of this segment is $\mu 2R\theta$, its acceleration is v^2/R, so $F = ma$ demands that

$$2\tau\theta = \mu 2R\theta v^2/R$$

or

$$\tau = \mu v^2.$$

The stress is τ/A and the volume density is $\rho = \mu/A$, so

$$\text{stress} = \rho v^2.$$

Relativistic limit on strength

The hoop velocity must be less than c. That is, the hoop must splinter at some $v < c$. That is, the ultimate tensile strength must satisfy

$$\text{UTS} < \rho c^2.$$

For a material with the density of diamond, 352 kg/m^3, what is the maximum possible ultimate tensile strength? Answer: 3.16×10^{19} N/m^2. For comparison, diamond actually has a UTS of 2.8×10^9 N/m^2, less than one ten-billionth of the maximum possible.

This is of course a hokey derivation, because it assumes classical mechanics in treating the forces on the hoop! I don't know how to improve it.

Relativistic limit on rigidity

When you study waves, you will find that the speed of sound waves in a solid is given by $v_p = \sqrt{Y/\rho}$. So the relativistic limit on Young's modulus is the same as the relativistic limit on ultimate tensile strength. Because the derivation of $v_p = \sqrt{Y/\rho}$ is classical, this argument suffers from the same hokeyness as the previous one.

Future work

You can see that this presentation is unsatisfactory. The subject is of astrophysical importance because our treatment of stellar evolution requires a better understanding of stresses and strains within neutron stars. Work is proceeding — under the name "relativistic elastodynamics" — to improve the unsatisfactory state of our understanding.

⟦See, for example, Gérard A. Maugin, *Continuum Mechanics Through the Twentieth Century: A Concise Historical Perspective* (Springer, Dordrecht, 2013) chapter 15: "Relativistic Continuum Mechanics: A 20th Century Adventure"; and Rémi Hakim, "Introduction to Relativistic Statistical Mechanics: Classical and Quantum" (World Scientific, Singapore, 2011).⟧