Oberlin College Physics 212, Fall 2021

Model Solutions to Assignment 2

Problem 2.4. Relativistic energy and momentum, I

A particle has relativistic energy equal to three times its rest energy. Find its resulting speed and momentum.
Answer:

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}} = 3mc^2, \tag{1}$$

So

$$\begin{aligned} \sqrt{1-(v/c)^2} &= \tfrac{1}{3} \\ 1-(v/c)^2 &= \tfrac{1}{9} \\ (v/c)^2 &= \tfrac{8}{9} \\ v &= \tfrac{\sqrt{8}}{3}c = 0.943\,c. \end{aligned} \tag{2}$$

The momentum is

$$p = \frac{mv}{\sqrt{1-(v/c)^2}} = 3mv = \sqrt{8}\,mc = 2.83\,mc. \tag{3}$$

How do these results change if the total energy is six times its rest energy? *Answer:*

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}} = 6mc^2, \tag{4}$$

So

$$\begin{aligned} \sqrt{1-(v/c)^2} &= \tfrac{1}{6} \\ 1-(v/c)^2 &= \tfrac{1}{36} \\ (v/c)^2 &= \tfrac{35}{36} \\ v &= \tfrac{\sqrt{35}}{6}c = 0.986\,c. \end{aligned} \tag{5}$$

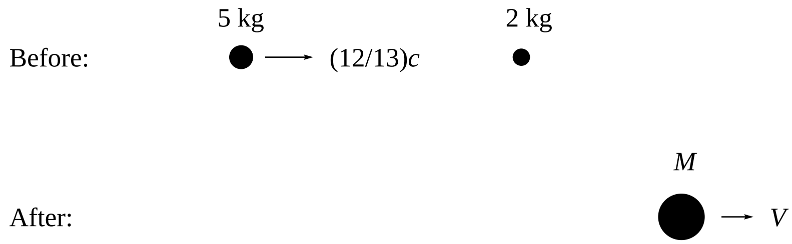
The momentum is

$$p = \frac{mv}{\sqrt{1-(v/c)^2}} = 6mv = \sqrt{35}\,mc = 5.92\,mc. \tag{6}$$

Thus the total energy doubles, the momentum more than doubles, but the velocity increases just a bit.

⟦*Grading:* 1 point each for setting up equations (1) and (4);
2 points each for equations (2), (3), (5), and (6).⟧

Problem 4.1. Sticky particles



Conserve energy:

$$\begin{aligned}
\frac{(5\text{ kg})c^2}{\sqrt{1-(\frac{12}{13})^2}} + (2\text{ kg})c^2 &= \frac{Mc^2}{\sqrt{1-(V/c)^2}} &\quad (7)\\
\frac{(5\text{ kg})c^2}{5/13} + (2\text{ kg})c^2 &= \frac{Mc^2}{\sqrt{1-(V/c)^2}} \\
(13\text{ kg})c^2 + (2\text{ kg})c^2 &= \frac{Mc^2}{\sqrt{1-(V/c)^2}} \\
15\text{ kg} &= \frac{M}{\sqrt{1-(V/c)^2}} &\quad (8)
\end{aligned}$$

Conserve momentum:

$$\begin{aligned}
\frac{(5\text{ kg})\frac{12}{13}c}{\sqrt{1-(\frac{12}{13})^2}} + (2\text{ kg})0 &= \frac{MV}{\sqrt{1-(V/c)^2}} &\quad (9)\\
\frac{(5\text{ kg})(12/13)c}{5/13} &= \frac{MV}{\sqrt{1-(V/c)^2}} \\
(12\text{ kg})c &= \frac{MV}{\sqrt{1-(V/c)^2}} &\quad (10)
\end{aligned}$$

These are two equations in two unknowns, and we solve them for M and V. Plug equation (8) into the right-hand side of equation (10) to find

$$(12\text{ kg})c = (15\text{ kg})V$$

whence

$$V = \tfrac{4}{5}c. \qquad (11)$$

(This speed is, of course, *less* than c, and also *less* than the incoming speed $\frac{12}{13}c$.)

Plug this value back into equation (8) to find

$$M = (15\text{ kg})\tfrac{3}{5} = 9\text{ kg}. \qquad (12)$$

A 5 kg object sticks to a 2 kg object to form a composite of mass 9 kg, not 7 kg. Energy is conserved, but "sum of masses of constituents" is not!

⟦*Grading:* 2 points for figure; 1 point each for equations (7), (8), (9), and (10); 2 points each for results (11) and (12). Special consideration: Problems 4.1 and 5.4 are the same problem. Under ideal conditions, students work the problem one week this way, then the next week they work it using the conserved invariant. That way they discover viscerally how much easier it is to use the conserved invariant. This year, the class sequence made both problems due the same week. A student solving both problems using the conserved invariant misses out on the visceral knowledge but has earned full credit.⟧

Problem 4.3: Sticky particles and the classical limit

Initial: ball of mass m and speed v plus ball of mass m and speed 0.
Final: ball of mass M and speed V.

a. Classically — conserve momentum: $mv = MV$ — conserve mass: $M = 2m$ — conclude $V = v/2$.

b. Relativistically — conserve momentum:

$$\frac{mv}{\sqrt{1-(v/c)^2}} = \frac{MV}{\sqrt{1-(V/c)^2}}. \tag{13}$$

Conserve energy:

$$\frac{mc^2}{\sqrt{1-(v/c)^2}} + mc^2 = \frac{Mc^2}{\sqrt{1-(V/c)^2}}. \tag{14}$$

Divide momentum equation by energy equation:

$$\frac{v/\sqrt{1-(v/c)^2}}{1/\sqrt{1-(v/c)^2}\,+1} = V$$

or

$$V = \frac{v}{1+\sqrt{1-(v/c)^2}}.$$

c. Classical limit: For $v \ll c$, $v/c \to 0$ and $V \to v/2$.

d. For any given v, the correct relativistic V is always larger than the classical $V = v/2$. (In particular, as $v \to c$ then the correct result is $V \to c$ not $V \to c/2$.) ⟦Note that all results obtained so far are independent of m.⟧

e. You can solve this through the algebra of solving equations (13) and (14) simultaneously (a gnarly proposition), or you can use the conserved invariant:

$$\begin{aligned} E^2 - (pc)^2 &= (Mc^2)^2 \\ \left(\frac{mc^2}{\sqrt{1-(v/c)^2}} + mc^2\right)^2 - \left(\frac{mvc}{\sqrt{1-(v/c)^2}}\right)^2 &= (Mc^2)^2 \\ \left(\frac{1}{\sqrt{1-(v/c)^2}} + 1\right)^2 - \left(\frac{v/c}{\sqrt{1-(v/c)^2}}\right)^2 &= \left(\frac{M}{m}\right)^2 \end{aligned}$$

$$
\begin{aligned}
\frac{1+2\sqrt{1-(v/c)^2}+[1-(v/c)^2]-(v/c)^2}{1-(v/c)^2} &= \left(\frac{M}{m}\right)^2 \\
2\left(\frac{1-(v/c)^2+\sqrt{1-(v/c)^2}}{1-(v/c)^2}\right) &= \left(\frac{M}{m}\right)^2 \\
m\sqrt{2}\sqrt{1+\frac{1}{\sqrt{1-(v/c)^2}}} &= M
\end{aligned}
$$

f. In the non-relativistic limit, $M \to 2m$.

g. In all cases, $M \geq 2m$.

⟦*Grading:* 1 point for each part, except 2 points for part b and 3 points for part e. A student making an error at, say, part b, and then propagating that error into future parts, has points deducted for the error at part b but has earned full credit for the latter parts (provided no errors are made in the propagation!).⟧

Problem 5.4: Sticky particles, II

The quantity $E^2-(pc)^2$ is invariant between frames and conserved through time. Initial values, in lab frame:

$$
\begin{aligned}
E &= \frac{m_1c^2}{\sqrt{1-(v/c)^2}} + m_2c^2 = \frac{(5\text{ kg})c^2}{5/13} + (2\text{ kg})c^2 = (15\text{ kg})c^2 \\
p &= \frac{m_1v}{\sqrt{1-(v/c)^2}} = \frac{(5\text{ kg})(12/13)c}{5/13} = (12\text{ kg})c \\
E^2-(pc)^2 &= (15\text{ kg})^2c^4 - (12\text{ kg})^2c^4 = (9\text{ kg})^2c^4
\end{aligned}
$$

Final values, in the lump's frame:

$$
\begin{aligned}
E &= Mc^2 \\
p &= 0 \\
E^2-(pc)^2 &= M^2c^4
\end{aligned}
$$

Using the conserved invariant, $M = 9$ kg. Then, in the lab frame,

$$
\frac{V}{c} = \frac{pc}{E} = \frac{(12\text{ kg})c^2}{(15\text{ kg})c^2} = \frac{4}{5}.
$$

⟦*Grading:* 1 points each for set up of initial lab E, initial lab p, initial lab $E^2-(pc)^2$, final lump E, final lump p, final lump $E^2-(pc)^2$; 2 points each for M and V/c. Error propagation as before.⟧