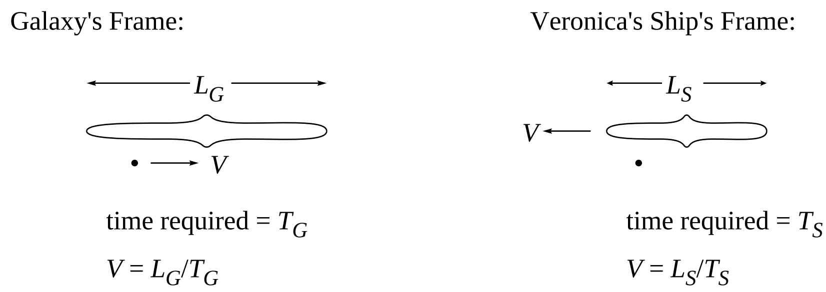
## Galactic Journey

*Step 1: Setup.* Sketch, define symbols, orient yourself. (In all physics, the first step in solving a problem is to make a sketch. But in relativity, the first step is often to make two sketches, one for each relevant reference frame.)




Our task: Given $L_G$ and $T_S$, find $V$. (The quantity $T_G$ is not given, so must be eliminated in any useful equation for $V$.)

*Step 2: Use some strategy to find $V$ in terms of the givens.* This model solution will use four different strategies, each one correct, and you can select a preference yourself.

*Strategy A: Time dilation.* Use

$$T_G = \frac{T_S}{\sqrt{1-(V/c)^2}}. \tag{1}$$

This gives

$$V = \frac{L_G}{T_G} = \sqrt{1-(V/c)^2}\,\frac{L_G}{T_S} \quad \text{resulting in} \quad V = \sqrt{1-(V/c)^2}\,\frac{L_G}{T_S}. \tag{2}$$

The right-hand equation involves only known quantities ($L_G$ and $T_S$) and the desired quantity ($V$).

We are done with physics. We need only solve the right-hand equation for $V$:

$$\begin{aligned}
(V/c)^2 &= [1-(V/c)^2]\left[\frac{L_G/c}{T_S}\right]^2 \\
(V/c)^2[T_S/(L_G/c)]^2 &= 1-(V/c)^2 \\
(V/c)^2\left\{1+[T_S/(L_G/c)]^2\right\} &= 1 \\
V/c &= \frac{1}{\sqrt{1+[T_S/(L_G/c)]^2}}
\end{aligned}$$

*Strategy B: Length contraction.* Use

$$L_S = \sqrt{1-(V/c)^2}\,L_G. \tag{3}$$

This gives

$$V = \frac{L_S}{T_S} = \sqrt{1-(V/c)^2}\,\frac{L_G}{T_S} \quad \text{resulting in} \quad V = \sqrt{1-(V/c)^2}\,\frac{L_G}{T_S}. \tag{4}$$

The right-hand equation involves only known quantities ($L_G$ and $T_S$) and the desired quantity ($V$).

We are done with physics. We need only solve the right-hand equation for $V$:

$$\begin{aligned}
(V/c)^2 &= [1-(V/c)^2]\left[\frac{L_G/c}{T_S}\right]^2 \\
(V/c)^2[T_S/(L_G/c)]^2 &= 1-(V/c)^2 \\
(V/c)^2\left\{1+[T_S/(L_G/c)]^2\right\} &= 1 \\
V/c &= \frac{1}{\sqrt{1+[T_S/(L_G/c)]^2}}
\end{aligned}$$

*Strategy C: Lorentz transformation.*

event 1: Veronica's ship departs from left edge of galaxy
event 2: Veronica's ship arrives at right edge of galaxy
$\Delta x = L_G \qquad \Delta x' = 0$
$\Delta t = T_G \qquad \Delta t' = T_S$

Goal: eliminate the unknown and undesired quantity $\Delta t = T_G$.

For any pair of events:

$$\Delta x' = \frac{\Delta x - V\Delta t}{\sqrt{1-(V/c)^2}}, \tag{5}$$

$$\Delta t' = \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}}. \tag{6}$$

Applying $\Delta x' = 0$ to equation (5) gives

$$\Delta t = \Delta x/V \tag{7}$$

(which is obvious anyway), and using this in equation (6) gives

$$\Delta t' = \frac{\Delta x/V - V\Delta x/c^2}{\sqrt{1-(V/c)^2}} = \frac{1/V - V/c^2}{\sqrt{1-(V/c)^2}}\Delta x. \tag{8}$$

Remembering that $\Delta x = L_G$ and $\Delta t' = T_S$ are both known quantities gives

$$T_S = \frac{1/V - V/c^2}{\sqrt{1-(V/c)^2}}L_G = \frac{1/(V/c) - V/c}{\sqrt{1-(V/c)^2}}L_G/c. \tag{9}$$

This equation involves only known quantities ($L_G$ and $T_S$) and the desired quantity ($V$).

We are done with physics. We need only solve the equation for $V$:

$$\begin{aligned}
[T_S/(L_G/c)] &= \frac{1/(V/c) - V/c}{\sqrt{1-(V/c)^2}} \\
[T_S/(L_G/c)] &= \frac{1}{V/c}\frac{1-(V/c)^2}{\sqrt{1-(V/c)^2}}
\end{aligned}$$

$$
\begin{aligned}
[T_S/(L_G/c)] &= \frac{1}{V/c}\sqrt{1-(V/c)^2} \\
[T_S/(L_G/c)]^2 &= \frac{1}{(V/c)^2}\{1-(V/c)^2\} \\
[T_S/(L_G/c)]^2 &= \frac{1}{(V/c)^2} - 1 \\
V/c &= \frac{1}{\sqrt{1+[T_S/(L_G/c)]^2}}
\end{aligned}
$$

*Strategy D: Interval.*

event 1: Veronica's ship departs from left edge of galaxy
event 2: Veronica's ship arrives at right edge of galaxy
$\Delta x = L_G \qquad \Delta x' = 0$
$\Delta t = T_G \qquad \Delta t' = T_S$

Goal: eliminate the unknown and undesired quantity $\Delta t = T_G$.

For any pair of events:

$$(c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2 - (\Delta x')^2. \tag{10}$$

But because

$$V = \Delta x/\Delta t \quad \text{we have} \quad \Delta t = \Delta x/V = L_G/V, \tag{11}$$

the interval between departure and arrival is

$$(L_G/(V/c))^2 - L_G^2 = (cT_S)^2 - 0^2 \tag{12}$$

or

$$-L_G^2[1 - 1/(V/c)^2] = (cT_S)^2. \tag{13}$$

This equation involves only known quantities ($L_G$ and $T_S$) and the desired quantity ($V$).

We are done with physics. We need only solve the equation for $V$:

$$
\begin{aligned}
1 - 1/(V/c)^2 &= -[T_S/(L_G/c)]^2 \\
1/(V/c)^2 &= 1 + [T_S/(L_G/c)]^2 \\
V/c &= \frac{1}{\sqrt{1+[T_S/(L_G/c)]^2}}
\end{aligned}
$$

*Step 3: Plug in numbers.* Regardless of strategy used, we find that

$$V/c = \frac{1}{\sqrt{1 + [T_S/(L_G/c)]^2}}. \tag{14}$$

Now, $T_S = 10$ years and $L_G = c(100,000 \text{ years})$, so

$$[T_S/(L_G/c)] = 10^{-4} \quad \text{and} \quad [T_S/(L_G/c)]^2 = 10^{-8}, \tag{15}$$

a dimensionless number much less than 1.

Remember that when $\epsilon \ll 1$,

$$(1+\epsilon)^n \approx 1 + n\epsilon. \tag{16}$$

Apply this to equation (14) with $\epsilon = [T_S/(L_G/c)]^2$ and $n = -1/2$ to find

$$V/c \approx 1 - \tfrac{1}{2}10^{-8} = 0.999\,999\,995. \tag{17}$$

*Step 4: Reflect.* If you want to fly across the galaxy while aging only 10 years, you can do it! But as this problem makes clear, your speed (in the galaxy's frame) will have to be just less than the speed of light.

(A) If you want to age even less than 10 years, just decrease $T_S$ in equation (14). You will need a speed even closer to the speed of light. In the limit that $T_S \to 0$, the speed required $V \to c$. A photon is produced at one edge of the galaxy and then absorbed at the far edge of the galaxy. In the galaxy's frame, this requires a time of 100,000 years. But in the photon's "frame" no time has elapsed at all: the photon is produced and then absorbed at the same instant. Is that utterly strange? No, because in the photon's "frame", it has traveled no distance at all: the photon is produced and then absorbed at the same place!

(B) Suppose you journey across the galaxy, aging 10 years. Once you get to the far edge you decide to return, at the same speed. During your homeward journey you again age 10 years, so you return home having aged 20 years. How much have your homebound friends aged? Just over 200,000 years!

⟦*Grading:* 3 points for setup: students don't need to use the same symbols I've used, of course, but for full credit they do need to set up rather than rush in. If they have only one figure they have earned 2 (rather than 3) points. If no figures they have earned 1 point. Most students will not separate the strategy and plug in numbers steps, and that's okay. There are 7 points for successfully finding the speed after setting up. The reflection is the most important step, but it's not required by the problem statement so no points are lost for lack of reflection.⟧