Fall semester, 2011 (revised version)
Oberlin College Physics 110

Notes for

Mechanics and Relativity

There is, in nature, perhaps nothing older than motion, concerning which books written by philosophers are neither few nor small; nevertheless I have discovered by experiment some properties of it which are worth knowing. . .

— Galileo Galilei
Two New Sciences (1638)

Important dates

- Homework due each Wednesday (unless there is an exam).
- First hour exam: Wednesday, 12 October 2011.
- Second hour exam: Wednesday, 16 November 2011.
- Lab graded in detail: Wednesday or Thursday, 2 or 3 November 2011.
- Final exam: Saturday, 17 December 2011; 2:00 to 4:00 PM (the time set by the registrar).

Class meetings

lectures	MW 9:00–9:50 AM	Wright 201	Mr. Styer
workshop A	MW 2:30–4:20 PM	Wright 107	Mr. Santos
workshop B	TTh 9:00–10:50 AM	Wright 107	Mr. Styer
workshop C	TTh 2:30–4:20 PM	Wright 107	Mr. Santos
informal Fridays	F 9:00–9:50 AM	Wright 201	

Tentative lecture and laboratory workshop schedule

⟦Next page⟧

7 September	What is time?
7–8 September	*Problem Solving*
12 September	Motion in one dimension
14 September	Special kinds of motion
14–15 September	*Bouncing Ball*
19 September	More on motion; Vectors
21 September	Motion in two and three dimensions
21–22 September	*Car Jump*
26 September	What causes motion? Forces
28 September	More on force
28–29 September	*Pendulum Challenge*
3 October	Newton's third law
5 October	Motion with non-constant force
5–6 October	*Terminal Velocity*
10 October	Impulse and momentum
12 October	**Exam**
12–13 October	*No lab*
17 October	Work and kinetic energy
19 October	Examples of work
19–20 October	*Carts and Energy*
24–28 October	**Fall break**
31 October	Potential energy and conservation of energy
2 November	Collisions
2–3 November	*Simple Harmonic Oscillator* [**graded in detail**]
7 November	Systems; Rotation
9 November	Rotational inertia; Angular momentum
9–10 November	*Car Collisions*
14 November	Elliptical orbits
16 November	**Exam**
16–17 November	*Rotation and Gyros*
21 November	**Relativity** — Speed of light; Time dilation
23 November	Length contraction; Relativity of simultaneity
23–24 November	**Thanksgiving**
28 November	Example: The hungry traveler
30 November	Lorentz transformation
30 Nov–1 Dec	*Special Relativity Paradoxes*
5 December	Force, momentum, and energy in relativity
7 December	General relativity
7–8 December	*Free-Wheeling Lab*
12 December	Twin paradox

Contents

Chapter 1

Introduction

Teachers:

Daniel F. Styer
Wright Laboratory room 215
775-8183, Dan.Styer@oberlin.edu
Office hours: Tuesday 2:30–3:30 PM, Friday 10:00–11:00 AM, and by appointment
Home telephone 281-1348 (2:30 PM to 9:00 PM only)

Aaron Santos
Wright Laboratory room 210
775-8566, Aaron.Santos@oberlin.edu
Office hours: Tuesday 10:00–11:00 AM and Thursday 9:00–10:00 AM

Course web page: `http://www.oberlin.edu/physics/dstyer/MechAndRel/`
I will post handouts, problem assignments, and model solutions here.

The word "physics" derives from the Greek word for nature. As such, the domain of physics stretches from atoms, molecules, and nuclei to baseballs, trees, and persons to planets, stars, and galaxies. Anyone who is interested in the universe, and in his or her own place in the universe, is interested in physics. This course introduces you to physics through the topics of mechanics (motion, force, momentum, and energy) and relativity (motion, force, momentum, and energy at very high speeds).

Course goals:

This course aims to broaden, deepen, and sharpen your scientific thinking skills. Such improvement cannot, however, be done in the abstract — it must be carried by a vehicle of specific topics and skills.

Knowledge: Introductory classical mechanics
Special relativity
Skills: Problem solving
Reasoning from observations and experiment
"Reading an equation"
Working with equipment and with phenomena

Course themes:

Applications — from atoms to automobiles to galaxies
Problem solving
Confront misconceptions
Confront the problems of everyday terms adopted as technical language
Mutually supporting qualitative and quantitative insight
Intellectual rigor — use reason, don't quote authority
Science is about nature, not about vocabulary or memorization

Outline of Questions and Topics

What is time?
Motion in one dimension

What causes motion?
Force in one dimension

Can this be generalized?
Motion and force in three dimensions

How can you measure the effect of a force?
Impulse and momentum

How else can you measure the effect of a force?
Work and energy

What are more applications?
Orbits, weightlessness, oscillators, waves, rigid bodies, gyroscopes

What happens if you move quite quickly?
Special relativity:
Time dilation, length contraction, relativity of simultaneity
Lorentz transformation
Relativistic momentum and energy

Format

This course contains both lectures and workshops. Each week, the entire class meets for two lectures (one hour each), and it breaks into small groups for two workshops (two hours each). The workshops include laboratory experiments, discussion, small group work, development of problem-solving skills, help with assigned homework, and other teaching that doesn't fit well into a large group format.

Approach

There are a lot of details in this course — you should *not* memorize them. Instead, you should be able to work with the information, to make inferences from the data, to solve problems that require an understanding of the material. In the words of Charles W. Misner:

> "The equation $F = ma$ is easy to memorize, hard to use, and even more difficult to understand."

Problem solving:

It will not take you long to notice that the problems and exams in this course — or any other physics course — exercise not only your knowledge of physics but also your skills in solving problems. One of the specific goals for this course is to teach not just about the content of mechanics and relativity, but also about problem solving. You will find many hints for honing your problem-solving skills in the books by Elby, Browne, and Pólya (on reserve — see below). The course web site includes tips for solving problems. And we will bring up suggestions in both lectures and workshops whenever the opportunity arises.

Reading an equation:

An equation may appear to be brief, and it may appear to be just a jumble of symbols. But there is a meaning, a story, behind every equation. One of the course goals is to help you uncover these meanings — to learn to "read an equation" in the same way that a connoisseur can "read a painting" to uncover meaning that lies beneath the surface, or that a white-water kayaker can "read a river" to discern dangerous and safe passages through a rapid before actually running the rapid. I sometimes call this "investing an equation with meaning". Regardless of your future life and career, you will find this to be a valuable skill. If you are faced in debate with an opponent who writes down an inscrutable equation and claims that this equation proves his point, you should then demand that your opponent describe in words its meaning — that he provide the story behind the equation. If your opponent can't do so, then it's as if he had used a five syllable word for effect — and he didn't know its meaning.

Hints for doing well in the course:

We recommend that you first do the readings, then attend the lectures, and then work on the problem assignments. More tips can be found through the course web site, but we cannot overemphasize that *we expect you to read the textbook.*

Readings

Required readings: (To be purchased.)

David Halliday, Robert Resnick, and Jearl Walker, *Fundamentals of Physics*, ninth edition (John Wiley and Sons, New York, 2011). [Referred to as "HRW". We will use chapters 1 through 11, plus a little bit of chapters 15 and 37. Either the standard or the extended version is okay. From what I can see, the eighth edition is also all right, although problem numbers might be different.]

Keep your workshop notes in a bound, quadrille-ruled lab book.

Supplemental readings: (The following books are on reserve in the science library. They are located on shelves along the south wall, not far to your right as you enter, near some comfy chairs to encourage browsing.)

Problem solving tips and techniques:

Andrew Elby, *The Portable TA: A Physics Problem Solving Guide* (Prentice Hall, 1998) [Oversize QC32.E56 1998]. Be sure to read the introduction (vol. I, page vii), the test taking tips (vol. II, page 327), and the advice on romance (vol. II, page ix).

Michael E. Browne, *Schaum's Outline of Physics for Engineering and Science* (McGraw Hill, 1999) [QC21.2.B77 1999].

(Some student find it profitable to purchase one or the other of the above two books.)

Sanjoy Mahajan, *Street-Fighting Mathematics: The Art of Educated Guessing and Opportunistic Problem Solving* (MIT Press, 2010). "In problem solving, as in street fighting, rules are for fools." Available through

`http://mitpress.mit.edu/books/full_pdfs/Street-Fighting_Mathematics.pdf`

George Pólya, *How To Solve It* (Princeton University Press, 1973) [Mudd QA11.P6 1973].

What is time?

Klaus Mainzer, *The Little Book of Time* (Copernicus, 2002) [QB209.M3513 2002].

Scientific American, special issue on time: September 2002. [Not on reserve.]

Galileo and the origin of mechanics:

Galileo Galilei, *Two New Sciences* (1638; Stillman Drake, translator, University of Wisconsin Press, 1974) [QC123.G13 1974]. When Galileo produced the new science of mechanics, he couldn't expect people to just take his word for it. Instead, he came up with ingenious, convincing — and funny — arguments that are still worth reading today.

I. Bernard Cohen, *The Birth of a New Physics* (W.W. Norton, 1985) [QC122.C6 1985]. The best brief description of the scientific setting in which Galileo worked.

Dava Sobel, *Galileo's Daughter* (Walker, 1999) [QB36.G2 S65 1999]. The best brief description of the social setting in which Galileo worked.

Physics in the everyday world:

Jearl Walker, *The Flying Circus of Physics* (John Wiley and Sons, 1977) [QC32.W2 1977]. Rainbows and barking sands, superballs and bicycles.

Barry Parker, *The Isaac Newton School of Driving: Physics and Your Car* (Johns Hopkins University Press, 2003) [QC125.2.P37 2003].

Nathan A. Unterman, *Amusement Park Physics* (J. Weston Walch, Portland, Maine, 2001) [Oversize QC32.U57 2001].

David W. Hafemeister, *Physics of Societal Issues: Calculations on National Security, Environment, and Energy* (Springer, 2007) [QC28.H25 2007].

Georg Hähner and Nicholas Spencer, "Rubbing and Scrubbing" *Physics Today,* volume 51, number 9, pages 22–27 (September 1998). A nice introduction to our understanding of friction, from ancient Egypt to present-day research. [Not on reserve.]

Special topics:

Felix Klein and Arnold Sommerfeld, *Über die Theorie des Kreisels* (B.G. Teubner, Leipzig, 1897–1910) [531K6722]. "On the theory of tops." Check this out to see how so much can be deduced from so little.

Various approaches:

Paul G. Hewitt, *Conceptual Physics* (Addison Wesley, San Francisco, 2002) [QC23.2.H488 2002] and *Conceptual Physics: Practicing Physics* (Addison Wesley, San Francisco, 2002) [QC23.2.H49 2002]. Interesting approach emphasizing topics rather than analysis.

Larry Gonick and Art Huffman, *The Cartoon Guide to Physics* (Harper Collins, New York, 1990) [QC24.5.G66 1991b]. A surprisingly effective summary.

Eric M. Rogers, *Physics for the Inquiring Mind* (Princeton University Press, 1960) [QC23.R68]. An older book, and not targeted specifically to audience of this course, but nevertheless soothing and worthwhile.

David Halliday, Robert Resnick, and Jearl Walker, *Fundamentals of Physics,* ninth edition (Wiley, New York, 2010) [QC21.3.H35 2011].

If you find yourself growing bored in this course, then dip into any of these books for more elaborate (or more idiosyncratic) treatments:

A.P. French, *Newtonian Mechanics* (Norton, 1971) [QC125.2.F74]. Particularly good on orbits.

A.P. French, *Vibrations and Waves* (Norton, 1971) [QC235.F74].

Charles Kittel, Walter D. Knight, and Malvin A. Ruderman, *Mechanics* (Berkeley physics course; McGraw-Hill, 1965) [530B455 vol. 1].

Daniel Kleppner and Robert J. Kolenkow, *An Introduction to Mechanics* (McGraw-Hill, 1973) [QA805.K62].

Richard P. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics*, volume 1 (Addison-Wesley, 1963) [530F438F vol. 1].

If, on the other hand, you find yourself getting lost in the course, then try these books for a more algorithmic point of view:

F.W. Sears, M.W. Zemansky, and H.D. Young, *University Physics,* sixth edition (Addison-Wesley, 1983) [QC21.2.S36 1983].

Daniel Kleppner and Norman Ramsey, *Quick Calculus: A Self-Teaching Guide* (Wiley, 1985) [QA303.K665 1985].

Relativity has its own raft of books:

D.F. Styer, *Relativity for the Questioning Mind* (Johns Hopkins University Press, 2011) [QC173.55.S79 2011]. At a different level from this course but (as you might guess from the author's name) with an approach similar to the one I will use.

A.P. French, *Special Relativity* (Norton, 1968) [530.11F887S].

N. David Mermin, *Space and Time in Special Relativity* (McGraw-Hill, 1968) [QC6.M367 1989].

Edwin F. Taylor and John Archibald Wheeler, *Spacetime Physics,* second edition (W.H. Freeman, 1992) [QC173.65.T37 1992].

Robert Resnick, *Introduction to Special Relativity* (John Wiley, 1968) [530.11R312I].

Jones Hammond Smith, *Introduction to Special Relativity* (Benjamin, 1965) [530.11Sm61I].

John B. Kogut, *Introduction to Relativity* (Harcourt, 2001) [QC173.55.K64 2001].

Yuan Chung Chang, *Special Relativity and its Experimental Foundations* (World Scientific, Singapore, 1997) [QC173.65.C465 1997].

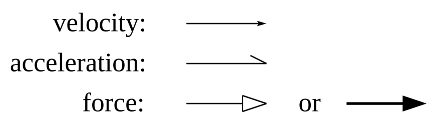
James B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addison Wesley, 2003) [QC173.6.H38 2003]. Nice discussion of gravitational time dilation on page 116.

There are a lot of supplemental readings here and I don't expect you to read all of them, or even any of them! Many of these topics could make good winter term projects. . . either of us would be happy to sponsor winter term projects inspired through this course.

Nuts and Bolts

Background: This course assumes a knowledge of the calculus at the level of Mathematics 133: *Calculus I: Limits, Continuity, Differentiation, Integration, and Applications* or the equivalent. If you lack this mathematical background then you will find this course to be excruciatingly difficult: you should either put off the course for a year while you study calculus or else enroll instead in the algebra/trigonometry-based course Physics 103: *Elementary Physics.* (It is also a good idea for you to currently be taking Mathematics 134: *Calculus II: Special Functions, Integration Techniques, and Power Series.*)

Arrowhead conventions: This course provides many occasions to draw different vector quantities. I will usually distinguish them by drawing a velocity with an open arrowhead, an acceleration with a one-sided arrowhead, and a force with a closed arrowhead.



Course assignments:

Readings from texts and from Galileo's *Two New Sciences*.
Weekly problem assignments.
Workshops.
Two one-hour exams and one two-hour final exam.
"Minute papers": You must submit to me a written reaction at the end of each lecture.

Problem assignments: The problem assignments in this course are not a dry appendage designed to keep you indoors on sunny days. Instead, the problems are central to your learning in the course. Problem solving is a more active, and hence more effective, way to learn than reading text or listening to lecture. Problem assignments will be posted on the course web site every Wednesday, and are due at the beginning of class the following Wednesday unless there is an exam. My model solutions will be posted at the end of this class, so late assignments cannot usually be accepted. (I may make an exception in the case of a medical or family emergency, but in most cases it is to your advantage to move on to the next assignment rather than to let old work pile up.) In writing your solutions, do *not* just write down the final answer. Show your reasoning and your intermediate steps. Describe (in words) the thought that went into your work as well as describing (in equations) the mathematical manipulations involved.

I encourage you to collaborate or to seek printed help in working the problems, but the final write-up must be entirely your own: you may not copy word for word or equation for equation. When you do obtain outside help you must acknowledge it. (E.g. "By integrating HRW equation (6-3) I find that. . . " or "Employing the substitution $u = \sin(x)$ (suggested by Carol Hall). . . " or even "In working these problems I benefited from discussions with Mike Fisher and John Silsbee.") Such an acknowledgement will never lower your grade; it is required as a simple matter of intellectual fairness. Each assignment will be graded by a student grader working under my close supervision.

Workshops: You are required to attend all your workshops. For the most part, the Monday or Tuesday workshop focuses on problem solving and conceptual understanding, while the Wednesday or Thursday workshop (the "lab workshop") focuses on interacting with phenomena and conceptual understanding. Many of the lab workshops come with "pre-lab warm-up questions" which must be answered through this course's BlackBoard site to your workshop instructor, who will grade them on a "fail-pass-super" basis. At the end of each lab, take your lab notebook to your instructor, who will take a quick look at it and grade five facets of the notebook, again on a three-point basis. For one lab — the "simple harmonic oscillator" lab on 3–4 November — you will turn in your lab notebook and your instructor will grade it in detail. Your lab grade is compounded from 1/3 pre-lab questions, 1/3 quick grading on every lab, and 1/3 on the lab graded in detail.

Exams: There will be two one-hour exams and one two-hour final exam. All exams are in-class. I will drop the lowest hour's worth of exam score in determining your grade (i.e. either the score of one hour exam or half the score of the final). No collaboration is permitted in working the exams. You may consult your textbook (HRW) and your own notes that fit on both sides of one $8\frac{1}{2}$ by 11 inch page of paper, but no other material. Calculators are permitted and encouraged. Exam questions will come from lecture, workshop, or problem assignment topics — I will not quiz you on obscure points deep in the text that I didn't emphasize. Before each exam I will distribute a topics list and a sample exam to give you an idea of what to expect.

At each exam you're allowed to bring your text and one sheet of paper with your own notes. (You may also, of course, make notes in the margins of your text.) What are the reasons for these rules?

1. I doubt that you'll use your notes, but the process of making up your notes — of deciding which ideas are the most important and the most useful — can be very helpful in giving you a clear overview of what's transpired in the course.

2. I doubt that you'll use your textbook, but the fact that you *can* bring it emphasizes that you're not supposed to memorize equations — you're supposed to work with and apply the ideas.

Minute papers: I will end every lecture a minute or two early so that you can write a brief (one- or two-sentence) reaction to the state of your knowledge concerning this course. Write this reaction, and your name, on a slip of paper and hand it in to me as you leave class. I will use these reactions to plan the next lecture and the future path of this course. Your most useful reaction would be a specific question: for example, "How can the net force acting on an object be directed toward the right while the object is moving toward the left?" Other possible reactions would be indications of general interest ("I'd like to learn more about the relation between oscillations and time-keeping.") or general questions ("Why should I care about this stuff, anyway?"). Please avoid questions of marginal relevance to this course ("How can I get that cute redhead in the second row to notice me?").

The ability to answer questions is an important skill. The ability to ask them is too. The problem assignments hone your answering skills, and the minute papers hone your asking skills.

Guest lectures: The department of physics and astronomy periodically invites visiting scientists to lecture at Oberlin. I will announce these visits in class. If you attend the guest lecture and submit to me a one-paragraph description through this course's BlackBoard site, you will be awarded 20 extra-credit problem-set points.

The purpose of the guest lectures is to broaden your horizons: to show you physics as it is done today and to present you with a viewpoint different from my own. You will not understand everything that the visiting speakers say...neither will I! One objective of the guest lectures is to show you how to get *something* out of a talk even when you don't understand *everything* in the talk.

Informal Fridays: From time to time I will announce additional, non-required, talks on Fridays at 9:00 AM in Wright 201. These informal talks are designed to introduce you to physics research and the Oberlin Department of Physics and Astronomy. If you attend one of these Friday talks and submit to me a one-paragraph description through this course's BlackBoard site, you will be awarded 20 extra-credit problem-set points.

Grading: Your final numerical grade will be the average compounded of two parts problem assignments, two parts exam, and one part workshop. On a 40-point scale, those with 40–33 points earn the grade "A", 32–27 points earn the grade "B", 26–20 points earn the grade "C", 19 or fewer points earn the grade "F".

Chapter 2

Classical Mechanics

2.1 Various Kinds of Clocks

pendulum ("grandfather") clock
quartz crystal oscillator (used today in most clocks, watches, computers, and cell phones)
spring-driven balance wheel watch (the kind that needs to be wound)
atomic clock
fountain atomic clock (today's world standard)
motion of the sun in the sky (that is, the rotation of the earth) (the 1956–1967 world standard)
clepsydra – amount of water dripping from a tank through a standard-sized leak – such as the "Tower of the Winds" in Athens, built about 50 B.C.
tuning fork clock
hourglass
candle or incense burning clock
heartbeat (In many of his experiments regarding motion, Galileo used his own pulse as a clock.)
fingernail or hair growth
biological aging (e.g. wrinkling of skin, whitening of hair)
waterfall wearing away a cliff (F.M. Gradstein, J.G. Ogg, and A.G. Smith, *A Geologic Time Scale 2004* (Cambridge University Press, Cambridge, UK, 2004). In science library reference section QE508.G3956 2004.)

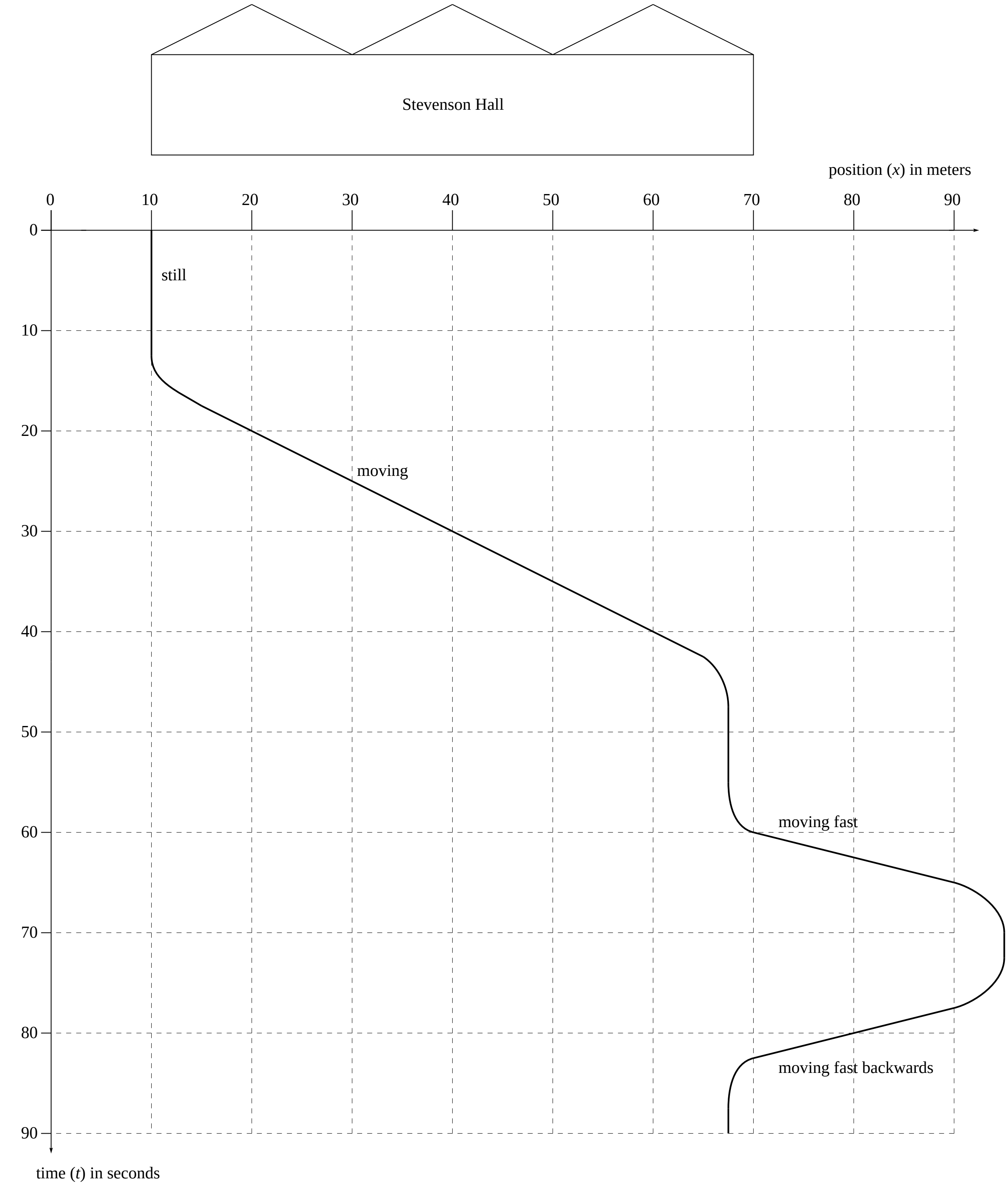
2.2 Describing Motion

To understand motion is to understand nature.

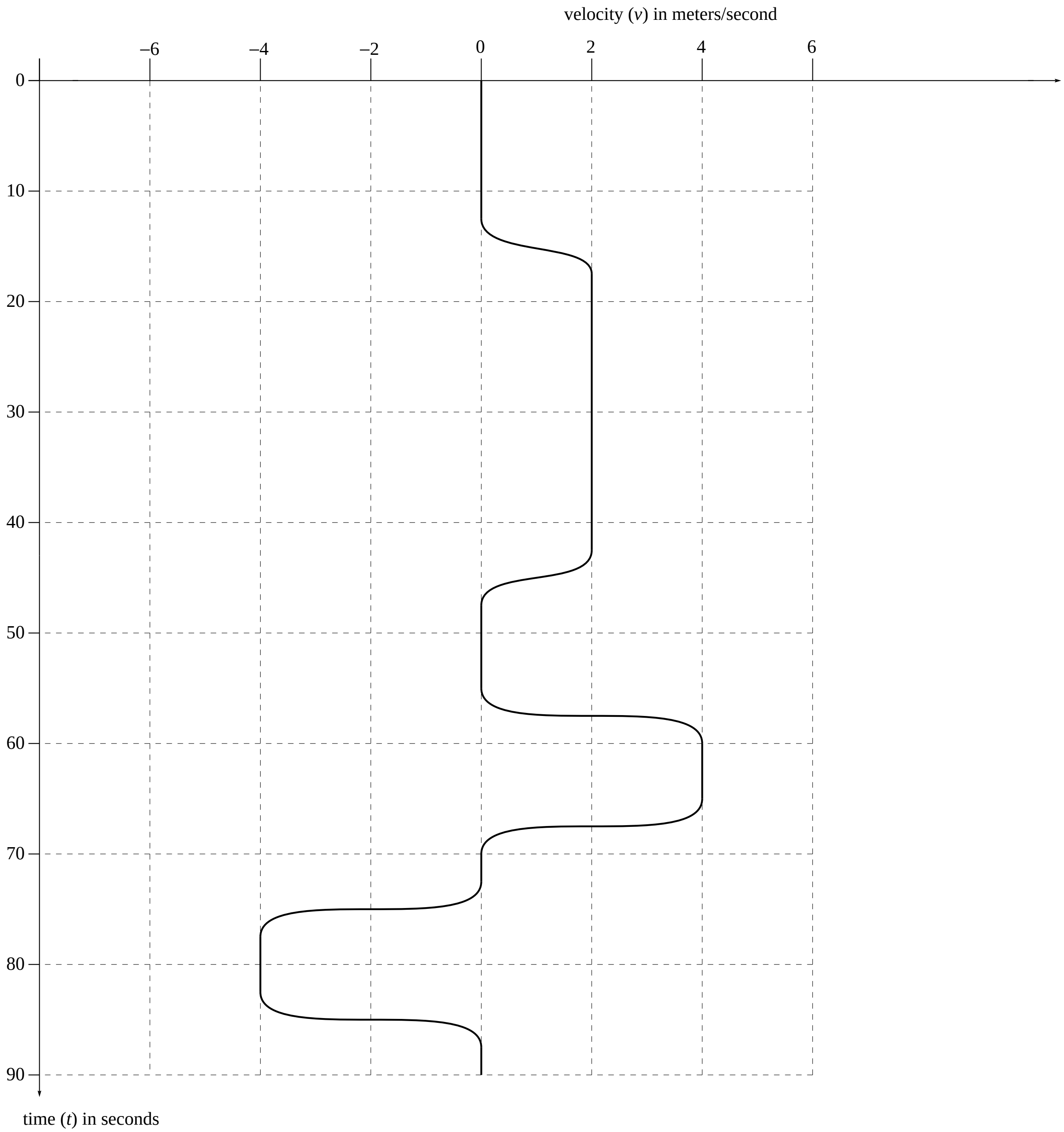
— Leonardo de Vinci

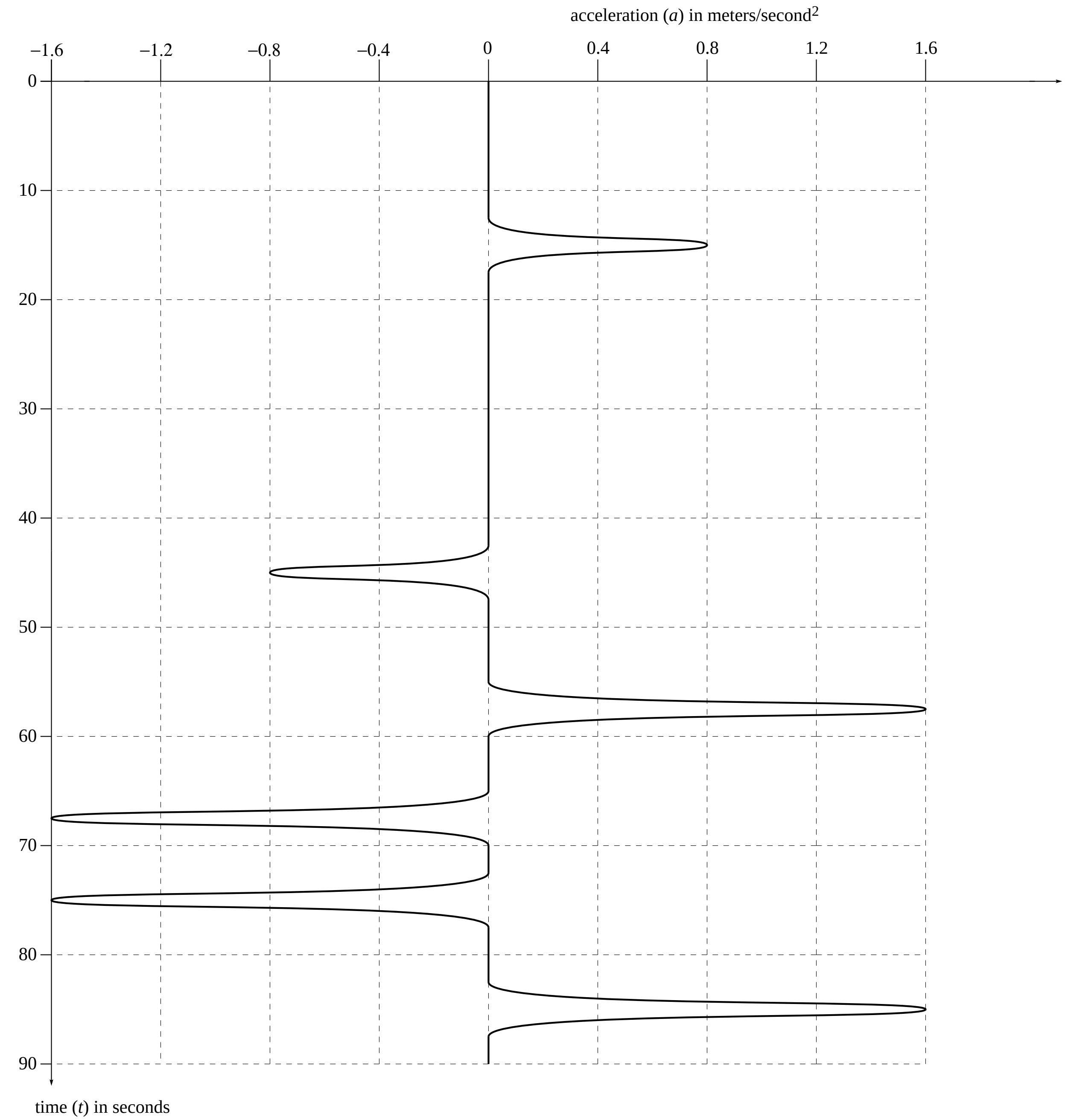
A bicyclist stands next to his bicycle, chatting with a friend near the north edge of Stevenson Hall. He mounts his bike and pedals south at a leisurely pace, until he sees another friend standing near the south edge of Stevenson Hall. He stops the bike to stand and chat with the second friend, then realizes he's late for class and pedals south rapidly. After a little while he realizes that he was wrong — today is Tuesday, not Monday, so he doesn't have a 9:00 AM class — so he turns around and pedals north rapidly to return to the conversation with his second friend.

This situation is summarized through the three graphs on the next three pages.



Stevenson Hall
position (x) in meters
0
10
20
30
40
50
60
70
80
90
still
moving
moving fast
moving fast backwards
time (t) in seconds


velocity (v) in meters/second
−6
−4
−2
0
2
4
6
0
10
20
30
40
50
60
70
80
90
time (t) in seconds


acceleration (a) in meters/second^2
−1.6
−1.2
−0.8
−0.4
0
0.4
0.8
1.2
1.6
0
10
20
30
40
50
60
70
80
90
time (t) in seconds

2.3 The Word "Acceleration"

It is a commonplace that one word can have several meanings, and that the meaning used in physics can differ dramatically from the meaning used in everyday speech. For example the word "acceleration" means one thing in physics and a different thing in everyday speech.[1] A car has a gas petal and a break petal. The gas petal is called "the accelerator" even though the break petal provides a greater magnitude of acceleration.

Many students find it difficult to accept that a ball, tossed vertically upward, experiences the same acceleration while going up, while going down, and at the instant between when it is going nowhere. Surely part of the reason for this difficulty is that they think of the ball as breaking while going up and accelerating only while going down.

2.4 Special Kinds of Motion

Motion with constant velocity

Suppose the velocity is a constant: $v(t) = v_0$. What is the acceleration? What is the position $x(t)$?

The acceleration is easy:

$$a(t) = \frac{dv}{dt} = 0. \tag{2.1}$$

The position is somewhat harder. Begin with

$$\frac{dx}{dt} = v_0. \tag{2.2}$$

Integrate each side with respect to time from some initial time t_i to some final time t_f

$$\int_{t_i}^{t_f} \frac{dx}{dt}\,dt = \int_{t_i}^{t_f} v_0\,dt. \tag{2.3}$$

The right hand side is easily integrated:

$$\int_{t_i}^{t_f} v_0\,dt = v_0 \int_{t_i}^{t_f} dt = v_0\ [t]_{t_i}^{t_f} = v_0[t_f - t_i]. \tag{2.4}$$

The left hand side is evaluated using the fundamental theorem of calculus, namely, that integration "undoes" differentiation:

$$\int_{t_i}^{t_f} \frac{dx}{dt}\,dt = [x(t)]_{t_i}^{t_f} = x(t_f) - x(t_i). \tag{2.5}$$

Equating these two gives

$$x(t_f) - x(t_i) = v_0[t_f - t_i] \tag{2.6}$$

or

$$x(t_f) = x(t_i) + v_0[t_f - t_i]. \tag{2.7}$$

[1] This was pointed out to me by Elizabeth W. Garbee, class of 2014.

It's conventional to set $t_i = 0\ldots$ we say "start the clock at the beginning of the time interval we're interested in." And since the end of the time interval could be any time at all, it's conventional to call t_f simply "t". Finally, it's conventional to call $x(0)$ by the name x_0. This gives

$$x(t) = x_0 + v_0 t \quad \text{for motion with constant velocity.} \tag{2.8}$$

Motion with constant acceleration

Suppose the acceleration is a constant: $a(t) = a_0$. Now we know that the velocity is *not* a constant — it will grow larger (when $a_0 > 0$) or smaller (when $a_0 < 0$) as time goes on. But how exactly will the velocity change? And what is the position $x(t)$?

Begin with the velocity. We know

$$a(t) = \frac{dv}{dt} = a_0. \tag{2.9}$$

Integrate each side with respect to time from some initial time t_i to some final time t_f

$$\int_{t_i}^{t_f} \frac{dv}{dt}\, dt = \int_{t_i}^{t_f} a_0\, dt. \tag{2.10}$$

The right hand side is easily integrated:

$$\int_{t_i}^{t_f} a_0\, dt = a_0 \int_{t_i}^{t_f} dt = a_0\ [t]_{t_i}^{t_f} = a_0[t_f - t_i]. \tag{2.11}$$

The left hand side is again evaluated using the fundamental theorem of calculus:

$$\int_{t_i}^{t_f} \frac{dv}{dt}\, dt = [v(t)]_{t_i}^{t_f} = v(t_f) - v(t_i). \tag{2.12}$$

Equating these two gives

$$v(t_f) = v(t_i) + a_0[t_f - t_i]. \tag{2.13}$$

Using the conventions described just above equation (2.8), plus the name $v(0) = v_0$, we have

$$v(t) = v_0 + a_0 t \quad \text{for motion at constant acceleration.} \tag{2.14}$$

Okay, that's the velocity. Now what about the position? Write the above equation as

$$\frac{dx}{dt} = v_0 + a_0 t. \tag{2.15}$$

Integrate each side with respect to time from some initial time t_i to some final time t_f

$$\int_{t_i}^{t_f} \frac{dx}{dt}\, dt = \int_{t_i}^{t_f} (v_0 + a_0 t)\, dt. \tag{2.16}$$

The right hand side is integrated as follows:

$$\begin{aligned}\int_{t_i}^{t_f}(v_0+a_0t)\,dt &= v_0\int_{t_i}^{t_f}dt+a_0\int_{t_i}^{t_f}t\,dt &(2.17)\\ &= v_0\,[t]_{t_i}^{t_f}+a_0[\tfrac{1}{2}t^2]_{t_i}^{t_f} &(2.18)\\ &= v_0[t_f-t_i]+\tfrac{1}{2}a_0[t_f^2-t_i^2]. &(2.19)\end{aligned}$$

Once again, the left hand side falls to the fundamental theorem of calculus:

$$\int_{t_i}^{t_f}\frac{dx}{dt}\,dt=[x(t)]_{t_i}^{t_f}=x(t_f)-x(t_i). \tag{2.20}$$

Equating these two gives

$$x(t_f)=x(t_i)+v_0[t_f-t_i]+\tfrac{1}{2}a_0[t_f^2-t_i^2]. \tag{2.21}$$

Using yet again the conventions described above equation (2.8), plus the name $x(0)=x_0$, we have

$$x(t)=x_0+v_0t+\tfrac{1}{2}a_0t^2 \quad \text{for motion at constant acceleration.} \tag{2.22}$$

There's one more thing we need to know: We have expressions for $v(t)$ and $x(t)$, but what about $v(x)$? We could just eliminate the variable t between the two equations (2.14) and (2.22), but there's a beautiful slick trick that is even easier. (I didn't think up this beauty: my teacher taught me. And his teacher taught him. I don't know who thought it up in the first place, but whoever it was deserves a metal.)

Let's look at acceleration — not just when the acceleration is constant, but in all cases:

$$\begin{aligned}a &= \frac{dv}{dt} \qquad [\![\ldots\text{use the chain rule to find}\ldots]\!] &(2.23)\\ &= \frac{dv}{dx}\frac{dx}{dt} \qquad [\![\ldots\text{then use } dx/dt=v \text{ to find}\ldots]\!] &(2.24)\\ &= v\frac{dv}{dx} &(2.25)\end{aligned}$$

Now we have an expression for acceleration involving $v(x)$... time doesn't enter the picture at all! This is called the $v(dv/dx)$ trick (pronounced "vee dee vee dee ex").

Applying this trick to the case of constant acceleration gives

$$v\frac{dv}{dx}=a_0. \tag{2.26}$$

Or (being a bit loose with the notation)

$$v\,dv=a_0\,dx. \tag{2.27}$$

Integrating each side (from v_i, x_i to v_f, x_f) gives

$$\begin{aligned}\int_{v_i}^{v_f}v\,dv &= \int_{x_i}^{x_f}a_0\,dx &(2.28)\\ [\tfrac{1}{2}v^2]_{v_i}^{v_f} &= a_0\,[x]_{x_i}^{x_f} &(2.29)\\ v_f^2-v_i^2 &= 2a_0[x_f-x_i]. &(2.30)\end{aligned}$$

Using our by-now-familiar conventions of $v_i=v_0$, $x_i=x_0$, $v_f=v$, and $x_f=x$, we have

$$v^2=v_0^2+2a_0(x-x_0) \quad \text{for motion at constant acceleration.} \tag{2.31}$$

2.5 Galileo on Falling Bodies

Galileo Galilei's *Discourses and Mathematical Demonstrations Concerning Two New Sciences* is perhaps the first book of modern science. It takes the form of a conversation between three gentlemen: Salviati, a supporter of Galileo; Simplicio, a defender of Aristotelian views; and Sagredo, a skeptical but open-minded layman with a deep interest in the world around him. The first new science concerns the strength of materials — what we would today call "materials science." The second new science is mechanics. But the conversation ranges widely over a host of topics: falling bodies, musical instruments,[2] the speed of light, the nature of infinity. It's important to realize that, while the thrust of the book is correct, Galileo also makes several errors.

Two New Sciences was written or dictated by Galileo in the final years of his life, while he was growing blind and was under house arrest by the Holy Office of the Inquisition for advocating the Copernican view of the solar system. It was published in Leyden in 1638. The following excerpt, from Stillman Drake's 1974 translation (pages 65–67), concerns falling bodies. It gives a good picture of Galileo's style of writing and of argumentation.

Simplicio. Aristotle... assumes that bodies differing in heaviness are moved in the same medium with unequal speeds, which maintain to one another the same ratio as their weights. Thus, for example, a body ten times as heavy as another, is moved ten times as fast...

Salviati. I seriously doubt that Aristotle ever tested whether it is true that two stones, one ten times as heavy as the other, both released at the same instant to fall from a height, say, of two hundred feet, differed so much in their speeds that upon the arrival of the larger stone upon the ground, the other would be found to have descended no more than twenty feet.

Simplicio. But it is seen from his words that he appears to have tested this, for he says "We see the heavier..." Now this "We see" suggests that he had made the experiment.

Sagredo. But I, Simplicio, who have made the test, assure you that a cannonball that weights one hundred pounds (or two hundred, or even more) does not anticipate by even one span the arrival on the ground of a musket ball of no more than half an ounce, both coming from a height of four hundred feet.

Salviati. But [even] without other experiences, by a short and conclusive demonstration, we can prove clearly that it is not true that a heavier body is moved more swiftly than another, less heavy, these being of the same material, and in a word, those of which Aristotle speaks. Tell me, Simplicio, whether you assume that for every heavy falling body there is a speed determined by nature such that this cannot be increased or diminished except by using force or opposing some impediment to it.

Simplicio. There can be no doubt that a given body in a given medium has an established speed determined by nature, which cannot be increased except by conferring on it some new impetus, nor diminished save by some impediment that retards it.

Salviati. Then if we had two bodies whose natural speeds were unequal, it is evident that were we to connect the slower to the faster, the latter would be partly retarded by the slower, and this would be partly speeded up by the faster. Do you not agree with me in this opinion?

Simplicio. It seems to me that this would undoubtedly follow.

[2] In this connection it is worth mentioning that Galileo's father, the composer and music theorist Vincenzo Galilei (*c.* 1525–1591), was one of the inventors of the art form known today as opera.

Salviati. But if this is so, and if it is also true that a large stone is moved with eight degrees of speed, for example, and a smaller one with four degrees, then joining both together, their composite will be moved with a speed less than eight degrees. But the two stones joined together make a larger stone than that first one which was moved with eight degrees of speed; therefore this greater stone is moved less swiftly than the lesser one. But this is contrary to your assumption. So you see how, from the supposition that the heavier body is moved more swiftly than the less heavy, I conclude that the heavier moves less swiftly.

Simplicio. I find myself in a tangle. . .

2.6 Galileo on Reference Frames

Galileo Galilei's *Dialogue Concerning the Two Chief World Systems — Ptolemaic and Copernican* is a bit earlier than *Two New Sciences*, and a bit funnier as well. As with *Two New Sciences* the dialog involves Salviati, Simplicio, and Sagredo.

The *Dialogue* was published in Florence in 1632. The following passage, from Stillman Drake's 1953 translation (pages 125, 186–188), treats the question: If the earth is in motion, then why don't we get left behind when we jump up?

Simplicio. Aristotle. . . strengthens [his arguments that the earth is stationary] with a fourth argument taken from experiments with heavy bodies which, falling from a height, go perpendicularly to the surface of the earth. Similarly, projectiles thrown vertically upward come down again perpendicularly by the same line, even though they have been thrown to immense height. These arguments are necessary proofs that their motion is toward the center of the earth, which, without moving in the least, awaits and receives them.. . .

Salviati. For a final indication of the nullity of the experiments brought forth, this seems to me the place to show you a way to test them all very easily. Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though there is no doubt that when the ship is standing still everything must happen this way), have the ship proceed with any speed you like, so long as that motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping towards the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and the flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship. . .

Sagredo. Although it did not occur to me to put these observations to test when I was voyaging, I am sure that they would take place in the way you describe. In conformation of this I remember having often found myself in my cabin wondering whether the ship was moving or standing still; and sometimes at a whim I have supposed it going one way when its motion was the opposite.

2.7 Galileo on Names

Another excerpt from Galileo's *Dialogue Concerning the Two Chief World Systems — Ptolemaic and Copernican*, Stillman Drake's 1953 translation (page 234).

Salviati. . . . I shall [be overjoyed] if [anyone] can teach me what it is that moves earthly things downward.

Simplicio. The cause of this effect is well known; everybody is aware that it is gravity.

Salviati. You are wrong, Simplicio; what you ought to say is that that everyone knows that it is called "gravity." What I am asking you for is not the name of the thing, but its essence, of which essence you know not a bit more than you know about the essence of whatever moves the stars around. I accept the name which has been attached to it and which has been made a familiar household world by the continual experience that we have of it daily. But we do not really understand what principle or what force it is that moves stones downward, any more than we understand what moves them upward after they leave the thrower's hand, or what moves the moon around.

2.8 What Causes Motions? Forces

A force is a push or a pull.

Here in the physics department, a force does not necessarily come through a human or a living agent (although this *is* true of the "moral force" that they speak of in the religion department). In fact, the physics-style force does not necessarily involve contact! If I hold up a brick, then the Earth's gravity pulls down on the brick, even though the Earth and the brick aren't in contact. The evidence that the brick is pulled down is that if I drop the brick it falls.

But what if I don't drop the brick — suppose that instead I set it on a pillow. Now the brick rests on the pillow. Even though gravity pulls it, the brick doesn't fall. Why not? Because the pillow pushes up on the brick. You can see that the pillow pushes, because it compresses when I place the brick on it. (We say that the pillow exerts an upward force on the brick that precisely cancels the downward force of gravity. We say this even though the term "exerts" suggests that the pillow is huffing and puffing and getting tired like a weight lifter. The physics term "exerts a force" differs considerably from the more familiar athletics term of "exertion".)

But what if I place the brick on a table top? In this case the table pushes up on the brick — but that seems wrong because the table, unlike the pillow, doesn't seem to compress. In fact, the table *does* compress when a brick is placed on it, but the compression is very small. If we were to put one thousand bricks on a table we would notice a sag. Both the pillow and the table compress, but the table is much stiffer than the pillow so its compression is barely noticeable to the naked eye.

Building a force meter: an operational definition of force

This basic idea — force as a push or a pull — needs to be refined and made quantitative. Just as we refined the qualitative idea of "time" by building clocks, so we will refine the qualitative idea of "force" by building force meters.

One way to measure forces is through rubber bands. I can loop a rubber band around my two thumbs, and place the thumbs ten inches apart. Now I can feel the rubber band pulling my two thumbs together. I could define this (operational definition) to be a force of one pound. To obtain a force of two pounds, I would stretch two identical rubber bands to a length of ten inches.

This is not just hypothetical. When my son wore dental braces, they were held in with "Tru-Force" brand orthodontic elastics (the technical term for rubber bands) that were calibrated to exert a force of 3.2 ounces when stretched to a length of 13/16 inch.

Now, rubber bands are not the most accurate force meters. For one thing, they tend to snap when they get old. For another, they're great at measuring pulls but not so great at measuring pushes.[3] More accurate force meters are made by springs: you can use delicate springs for measuring small forces, and stiff springs for measuring large forces. Your grocery store's spring scale is just a spring attached to a dial which measures the length of the spring.

At the moment, the most accurate force meter in the world is a device called the "proving ring". It is good for measuring both pushes and pulls. You can buy very accurate proving rings from the National Institute for Standards and Technology, just as you can buy very accurate kilogram masses from them. And for the same reason that NIST keeps its master kilogram in a vault where it can't rust or get dusty or be stolen, so it keeps its master proving ring in a similar vault. (A good discussion of proving rings, with photos, is at http://www.nist.gov/mel/mmd/mf/provingring.cfm.)

⟦An aside: In the physics laboratory the pound is a unit of force, even though in the grocery store the pound is a unit of quantity. This is because the force of gravity on a bunch of bananas is proportional to the quantity of bananas. Here on the surface of the Earth, that proportionality is used for commerce. If we did commerce on the surface of the Moon as well, we'd have to get used to the idea that a bunch of bananas that

[3]A good way to measure pushes is by pushing through a pillow. . . we've already seen that a pillow compresses in order to exert a push.

weighed three pounds on Earth would weigh half a pound on the Moon. The groceries we buy should really be doled out in kilograms rather than pounds, but I'm not going to get upset because I don't intend to take my bananas to the Moon. (The very most precise and pedantic people refer to this usage of "the pound" as the "the pound-force" and abbreviate it not as "lb" but as "lbf". This helps distinguish the pound as a unit of force from the pound as a unit of quantity from the pound as a British bank note.)

Here's an analogy: distances should be measured in miles or kilometers, but I commonly hear expressions like "Pittsburgh is three hours from Cleveland", meaning three hours of driving time. This statement is true when made in a bus station, but in an airport one frequently hears that "Pittsburgh is twenty minutes from Cleveland".]]

How does force affect motion?

Now that we know how to measure force, we can ask how force affects motion. Does force relate to velocity? To acceleration? To jerkiness? To some combination such as $5v + 7a$?

Your first thought is that force causes velocity. After all, it's natural to say "If you push on something, it moves." But if you push a toy car on a table, it keeps on moving even after your hand stops pushing it. Furthermore, if the toy car is already moving, you have to push (backwards) on it to get it to stop. (Or else friction needs to push on it to get it to stop.)

In fact, as we'll verify quantitatively in lab, force doesn't cause velocity, it causes acceleration.

Common: "If you push on something, it moves."
Correct: "If you push on something, its motion changes."

Galileo knew that this fact violated common sense, so he came up with not just one but two arguments to back it up. First, he said, suppose I have a very smooth track shaped like a V. A ball is released on the left arm of the V two feet above the trough. The ball descends to the trough, then keeps on going up the other side until it's just a shade under two feet above the trough on the other side. Due to friction, it doesn't get all the way up to the two-foot mark, but the smoother you make the track the closer it will come.

Now suppose the right arm were not symmetric with the left, but that it went up with a shallower slope. Now the ball travels further along the right arm, but it still ends up just a shade below the two foot mark.

Finally, suppose the right arm doesn't slope upward at all, but just extends horizontally. The ball then would never be able to go up to the two-foot level where it started, and therefore it would go forever. If there is no force (including no frictional force) then there is no change in motion, but there certainly can be motion. Force is not needed to keep an object moving.

Here's Galileo's second argument. Suppose I toss a ball directly upwards. It flies straight up into the air, then comes down right into my waiting hand. There is no horizontal force, and no horizontal motion.

Now suppose I try the same experiment while inside a jetliner moving at 400 mph. (Galileo actually invoked a large sailing ship, but you're more likely to have traveled in a jetliner than in a large ship.) While

I'm inside the cabin, I toss a ball directly upwards. It comes down right into my waiting hand. (It does not somehow get "stuck in the Earth's reference frame", in which case it would be moving backwards at 400 mph in the jetliner's frame!) In this case there is still no horizontal force, but there is horizontal motion at 400 mph.

How does force affect motion? — Quantitative

You can do experiments pulling objects using springs and rubber bands to measure the force exerted on the objects, and then measuring their acceleration. If you're doing these experiments on the Earth's surface, where gravity is present, it's easiest to exert the force horizontally and to measure the horizontal acceleration — this way you don't get confused by the gravitational force. And, like Galileo, do your best to make smooth surfaces so that extraneous friction forces don't confuse you. What do you find?

If a brick is pulled with one rubber band, stretched to five inches, and then pulled with two identical rubber bands, each stretched to five inches, then the second experiment results in exactly twice the acceleration. That's right: if we double the force we double the acceleration; if we triple the force we triple the acceleration. In general, the brick's acceleration is directly proportional to the force which, exerted on the brick, causes that acceleration:

$$a \propto F. \tag{2.32}$$

(It certainly makes sense that a should increase with F, but these experiments show more: that a increases linearly with F.)

So, what's the constant of proportionality? It's going to be different for different objects. Here's a neat argument about how the mass of the object will affect the acceleration.[4]

Suppose I pull a 3 kg brick with a rubber band stretched out to measure 2 pounds. The brick accelerates at, it turns out, 2.9 m/s^2.

Now suppose I pull two of these bricks adjacent to each other, each with its own rubber band exerting 2 pounds. Of course each brick accelerates at 2.9 m/s^2, so the two bricks stay right beside each other.

Finally, I do the same experiment but this time with a tiny amount of glue connecting the two bricks. Of course, I still have the same acceleration. But in this last case I have not two independent bricks, but one double brick of mass 6 kg pulled by two rubber bands for a total force of 4 pounds. Yet the acceleration must be 2.9 m/s^2. In other words, if we double both the applied force and the mass, the acceleration is unchanged. Acceleration must therefore be proportional to the ratio

$$a \propto \frac{F}{m}. \tag{2.33}$$

So we've seen that the constant of proportionality in equation (2.32) depends on the mass of the object. What does the constant of proportionality in equation (2.33) depend on?

[4]I thought that this argument was created by Galileo, but I've just checked a historical reference and I see that I'm wrong. I'm not sure who created it.

We can try various experiments: Does it depend on the object's color? No. On its height above sea level? No. On its chemical composition? (That is, does a kilogram of sulfur respond to force in the same way as a kilogram of uranium? A kilogram of hydrogen?) No. Despite the fact that sulfur, uranium, and hydrogen have very different hardnesses, strengths, densities, lusters, and so forth, they all respond to force in exactly the same way. Does it depend on the object's consciousness or vitality? No. A kilogram of stone behaves the same (in this respect) as a kilogram of living potted plant which behaves the same as a kilogram of sleeping squirrel which behaves the same as a kilogram of awakened squirrel. In fact, all experiments[5] show that the constant of proportionality in equation (2.33) doesn't depend on anything else. When measured, this constant is found to be

$$a = k\frac{F}{m} \quad \text{where} \quad k = 4.448\ \frac{\text{m}}{\text{s}^2}\frac{\text{kg}}{\text{lb}}. \tag{2.34}$$

A second operational definition of force

In equation (2.34), we have three distinct ways to measure a, m, and F. This is the only way we could establish the result as a physical law and measure the constant k. However, once the law is established, we can use it to make a second operational definition of force.

The first operational definition embodies the qualitative idea of "force is a push or pull". Force is measured through a stretched spring: "A one pound force is the force exerted by a standard spring stretched to the one-pound mark." In this scheme, the constant k is measured.

The second operational definition embodies the qualitative idea of "force causes a change in motion". Force is measured through the amount of acceleration it causes: "A one pound force is the force that causes a one kilogram object to accelerate at 4.448 m/s^2." In this scheme the constant k is part of the definition of force, so this constant is defined rather than measured.

Well, if k is going to be defined, why not define it to have a convenient value? The convention is to define k to be 1, and in this case force is measured not in the unit of pounds, but in the unit of newtons, where a newton is a kg·m/s^2. The two units of force, newtons and pounds, are thus related through

$$1\ \text{lb} = 4.448\ \text{N}. \tag{2.35}$$

Let's review the strategy: (1) Make an independent operational definition of a new quantity. (2) Use that definition to discover a new law of nature. (3) Use that law to make a second operational definition of the quantity.

This three-part strategy is used all the time. You might think that at the end we could breath a sigh of relief and throw out all our standard springs, relieved that we no longer have to protect them from rust (and thievery). But in fact it's a good idea to keep the old standards around, just in case it happens that the law of nature you discovered is not quite right.

[5] In spring 2007, Jens Gundlach and his colleagues at the University of Washington tested this principle at the precision of 5×10^{-14} m/s^2. See <http://www.aip.org/pnu/2007/split/819-1.html>.

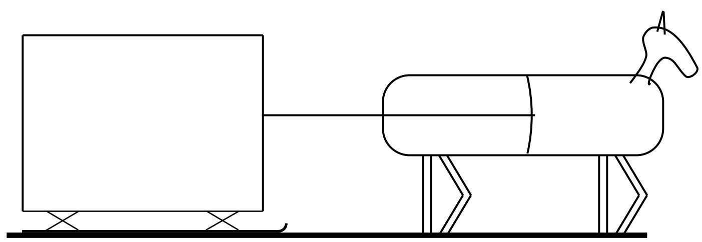
For example, the "$a = F/m$" law turns out to be just a bit off — it turns out that while the proportionally constant in equation (2.33) doesn't depend on the color, height, vitality, or composition of the object under acceleration, it does depend on its velocity. We now believe that the correct law is

$$a = \left(\sqrt{1 - v^2/c^2}\right)^3 \frac{F}{m},$$

where c is the speed of light.[6] As far as we know, there are no exceptions to this revised law (although its interpretation is delicate in quantum mechanical situations). Still, I wouldn't bet that we'll know of no exceptions 100 years from now. This shows that while $a = F/m$ can be used to provide a definition of force, that doesn't mean we've defined away a law of nature. Nature still has the upper hand.

2.9 The Horse and the Sled

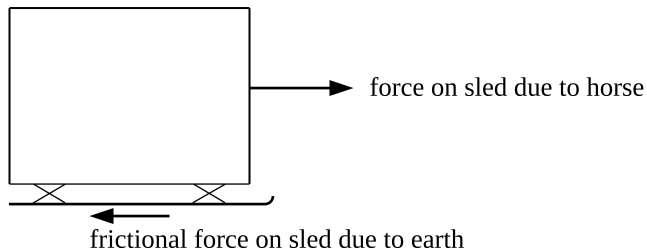
A horse pulls a sled down the road. The force exerted on the sled by the horse is equal and opposite to the force exerted on the horse by the sled. (I will use "horse" to mean "the combination of horse plus harness.") How can the sled accelerate if the two forces cancel out to zero?



The first step in resolving this paradox is (as usual) to sketch the situation. (In this discussion I will only treat horizontal forces, because the treatment of vertical forces and motions is straightforward.) This makes us realize that two horizontal forces act on the sled: the forward force due to the horse, plus a rearward frictional force due to the earth. (The frictional force actually acts all along the sled's runner — in the diagram I draw only a single force arrow because it's time consuming to draw an infinite number of little arrows.)

[6]Did I lie back above equation (2.34) when I said "all experiments show"? Yes, I lied. I couldn't think of any other way to put you into the right mood.

free body diagram of sled:



The force on the horse due to the sled *does not* act on the sled (it acts on the horse).

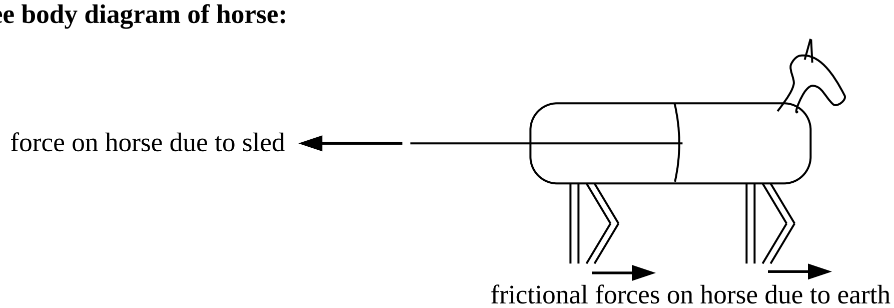
The force on the horse due to the sled *does not* appear in this free body diagram (it appears in the horse's free body diagram).

The force on the horse due to the sled *does not* enter into the sum of forces $\sum \vec{F}$ acting on the sled through $\sum \vec{F}_{\text{on sled}} = m_{\text{sled}}\vec{a}_{\text{sled}}$ (instead it enters into the sum of forces acting on the horse: $\sum \vec{F}_{\text{on horse}} = m_{\text{horse}}\vec{a}_{\text{horse}}$).

The acceleration is forward, because the force on the sled due to the horse is larger than the frictional force on the sled due to the earth.

We have resolved the horse-sled paradox. Nevertheless there are some interesting features that arise through examining the free body diagram of the horse.

free body diagram of horse:



As promised, the force on the horse due to the sled appears in this free body diagram, and it is equal and opposite to the force on the sled due to the horse that appears in the sled's free body diagram. And there are also frictional (contact) forces on the horse due to the earth. The forward forces (due to the earth) are greater than the rearward force (due to the sled), so the horse accelerates forward.

Q: How can the force of friction point forward? Friction always opposes motion!

A: No. Friction doesn't oppose motion, friction opposes slippage. Suppose a stationary horse places his hoof, then attempts to draw it backwards. If there were no friction, then the hoof, being drawn backwards, would indeed slip backwards and the horse as a whole would not move. However friction opposes this attempted backwards slippage — the friction pushes forwards.

Don't believe it? Then take a few steps and examine how your own footfalls behave. (Why must you walk more carefully on ice, where there is less friction?) Better still, sit in a wheeled chair and pull yourself forward around the room using only your feet.

Q: You've ignored all the forces within the horse, such as the force by the leg muscle on the leg bone, or the force by the leg bone on the hoof.

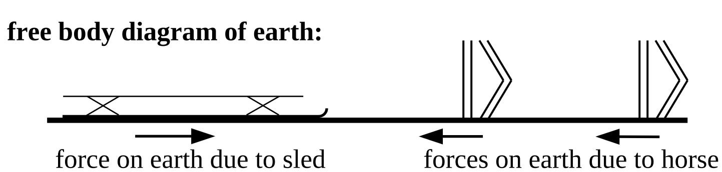
A: Yes I have, because the force by the leg muscle on the leg bone is equal and opposite to the force by the leg bone on the leg muscle. These internal forces are necessary so that the horse can "attempt to drag its hoof backwards." But internal forces by themselves cancel out in pairs and cannot accelerate the horse.

Q: How can the earth supply the (frictional) force that accelerates the horse? We all know that the energy required to accelerate the horse comes from the horse's muscles. That's why the horse gets tired.

A: We haven't yet studied energy, but when we do we'll find that, yes, the *energy* comes from the horse's muscles (and this energy is required for the horse to "attempt to drag its hoof backwards"), but the *force* comes from the earth.

Q: You've drawn the force on the sled due to the horse, and the force on the horse due to the sled — a third-law pair (or "an action-reaction pair"). But you've also drawn frictional forces: the force on the sled due to the earth and the forces on the horse due to the earth. Where are the reaction forces associated with these forces?

A: These reaction forces act upon the earth:



(In this diagram I've show a bit of the horse and sled in order to provide context. In most cases a free body diagram will only show the object *experiencing* the forces, not the objects *exerting* those same forces.)

Q: But the three forces acting on the earth aren't balanced: there's a net force to the left.

A: That's correct, and as a result the earth accelerates to the left. When the horse and sled accelerate to the right, the earth accelerates to the left. But because the mass of the earth is so much greater than the mass of the horse and sled, the acceleration of the earth is negligible.

Exercise: Draw all three free body diagrams near each other, and identify the third-law pairs. Note that the two members of a third-law pair always act on different objects.

2.10 Newton's Third Law

⟦Read this section *after* completing the HRW problem concerning "Blocks" and the additional problem concerning "A girl, a sled, and an ice-covered lake."⟧

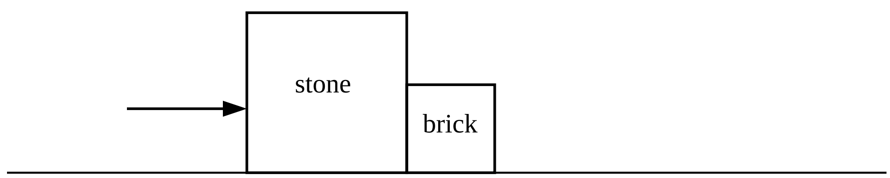
I hold an apple — there is a force of 1.2 newtons on the apple due to the earth ("the gravitational force"). North America is attracting the apple, Asia is attracting the apple, the iron core of the earth is attracting the apple. Every part of the huge planet earth is attracting the apple.

Now this tiny little apple is also attracting the earth — and the force on the earth due to this apple is exactly 1.2 newtons.

How can this little apple muster up as much force as the huge earth?

There are equal forces, but very different accelerations. In fact, the forces have to be equal so that the accelerations will be different! This is what we found in "A girl, a sled, and an ice-covered lake", although the ratio of masses in that case was not so huge.

I'll treat this again through an argument inspired by Galileo:



A hand applies a force of 36 N to a 3 kg stone, which in turn pushes a 1 kg brick. (Both stone and brick are resting on a frictionless surface.)

For the stone plus brick, $\sum F = ma$ gives 36 N = (4 kg) a, so $a = 9$ m/s^2.
For the brick alone, $\sum F = ma$ gives $F =$ (1 kg)(9 m/s^2) = 9 N, so the force on the brick due to the stone is 9 N.
Now, what's the force on the stone due to the brick?

If we believe Newton's third law, we say 9 N, so the net force on the stone is 36 N − 9 N = 27 N. Meanwhile the stone's mass is 3 kg, so its acceleration is $a =$ (27 N)/(3 kg) = 9 m/s^2, the same as before.

If we instead adopt the "bigger is better" viewpoint, then the force on the (big) stone due to the (small) brick is smaller than the force on the brick due to the stone... let's say 1/3 the size. Then the force on the stone due to the brick is 3 N, so the net force on the stone is 36 N − 3 N = 33 N. Meanwhile the stone's mass is 3 kg, so its acceleration is $a =$ (33 N)/(3 kg) = 11 m/s^2. But we've already found that the acceleration is 9 m/s^2, not 11 m/s^2. Thus the initial assumption must be wrong!

Conclusion. Equal and opposite forces doesn't mean:

Equal and opposite accelerations (e.g. apple and earth paradox).
Equal and opposite tiredness (e.g. if John pushes Mary in a wheelchair, John gets tired and Mary doesn't).
Equal and opposite damage (e.g. if a semi-truck collides with a Volkswagen, or with a mosquito).
Equal and opposite agency (or "causativeness", or "causation", or "moral force"). (e.g. In a barroom brawl, Angela hits Peggy and flattens her with a single blow. Peggy recovers and sues Angela for damages. Angela argues that she's innocent, because the force that Peggy exerted on Angela was equal to the force that Angela exerted on Peggy. This is correct physics, but it doesn't hold up in court: equal force doesn't imply equal damage or equal responsibility or equal guilt.)

Hint: Third-law pairs are always of the same *type*: The gravitational force of the earth on the apple is paired with the gravitational force of the apple on the earth. The contact force of a brick on the table is paired with the contact force of the table on that brick.

Misconception: Newton's third law is that if two objects, A and B, interact, the force on A due to B is the negative of the force on B due to A. Sometimes this is stated as "for every action there is an equal and opposite reaction". This is a very poor statement. It suggests that one force is the "action", which results in a second force, the "reaction". No. When the two objects interact, the two forces come into existence simultaneously, so one cannot be said to cause the other. When my sons were in middle school, they were assigned to classify forces into "action forces" and "reaction forces", an impossibility. If you had a similar task in middle school, please realize that it was inane and try not to let it influence your current thinking. *Do not confuse "action-reaction" with "cause-effect".*

2.11 The Meaning of the Word "Force"

⟦Read this section *after* completing the HRW problems concerning "Stationary salami" and "Blocks," and the additional problems concerning "A girl, a sled, and an ice-covered lake" and "Sliding salami."⟧

The word "force," like most English words, carries many meanings. According to the Oxford English Dictionary, there are 53 meanings of the noun "force"... and it's a verb, too! The everyday meaning of the word differs dramatically from the physics meaning of the word.

In the everyday sense of the word, we often think of force in terms of "domination" or "overpowering" — a direct command which you have no choice but to obey: "Timmy, I'm going to force you to eat your peas!" The physics sense of the word is quite different.

Examples:

(1) "My debate opponent initially held that taxes should be raised, but my persuasive arguments forced her to change her position." In debating, a force causes a change in position (i.e., a velocity) whereas in physics, a force causes a change in velocity (i.e., an acceleration). The “force of persuasive arguments” — like “the moral force” — can be exerted only by human beings, whereas “physics force” can be exerted by tables, chairs, blocks, and rocks as well.

(2) “Sam Jones tried to be an upstanding citizen, so he resisted overtures from organized crime. In the end, however, the Mafia forced him into the life of a criminal.” As used here, the term “force” doesn’t refer to something acting at a given instant, but rather to a pressure exerted over a period of time. While the physicist’s force acts at an instant and results in an acceleration, this everyday-style force acts over a period of time and results in a change of position (from “upstanding citizen” to “criminal”).

(3) “My employer forced me to move west from New York to Des Moines.” The employee didn’t have a choice, he *had* to move west. But the physics sense of the word carries no such implication: In circular motion, for instance, the force is perpendicular to the motion, so we might well have force towards the west, and at the same time motion towards the north. (If a thug with a gun told you “I am forcing you towards the west” and you moved north, this would be considered a violation of the command.[7]) Indeed, knowing the force alone doesn’t tell you the motion. . . you must also know the initial conditions.

(4) “During the Civil War, the army of the Union met the army of the Confederacy at Gettysburg. The force exerted by the Union on the Confederacy was greater than the force exerted in return, so the Confederate army had to move away and leave the Union army victorious.” This statement is perfectly correct military history. It shows once again that, in everyday contexts, force results in motion rather than in acceleration. And it demonstrates vividly that Newton’s third law does *not* apply to “force” as understood in military history.

(5) “The car swept by us with such force that we were pushed backwards on the sidewalk.” “The storm struck New England with hurricane-force winds of 70 miles per hour.” Goodness! The word “force” is used here to mean what a physicist means by “velocity”!

(6) “The most powerful force in the universe is love.” True in the vernacular sense of “power” and “force”, meaningless in the physics senses of those words.

(7) From *Star Wars: A New Hope*: “the Force. . . is an energy field” (Obi-Wan Kenobi). We haven’t yet talked about the physics meaning of the term “energy,” but when we do you’ll see that energy is very different from force.

When we investigated the concept of “time”, we started with the everyday meaning and made it more and more precise and rigorous — yet at the end of this refinement the word was simply a more precise version of our everyday concept.

But when we refined the concept of “force”, we started with “push or pull” and ended up with something rather different in qualitative content.

[7] Don’t try this at home.

I encourage you to develop your physical intuition, but be wary that intuition developed using the everyday meaning of the word force probably doesn't apply to its physics meaning.

More linguistic issues:

Avoid the phrase "the force produced by an acceleration"... $m\vec{a}$ is the result of a sum of forces. It is not a force!

Avoid the phrase "a brick has a force"... it experiences a force or it exerts a force.

Use the phrase "opposing force" only for two forces acting in opposite directions on the same object, not for third-law pairs! The two members of a third-law pair always act upon different objects, so they don't "oppose" each other.

2.12 Impulse, Momentum, Center of Mass

For a single particle:

The net impulse is $\vec{J} = \int_{t_i}^{t_f} \vec{F}_{\text{net}}\, dt$. This is defined over a time interval (from t_i to t_f).
The momentum is $\vec{p} = m\vec{v}$. This is defined at an instant.
These are related through $\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$.

For a system (of three particles):

The total mass is $M = m_1 + m_2 + m_3$.
The center of mass (or "mass-weighted average position") is

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}.$$

The net external impulse is

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}_{1,\text{ext}} + \vec{F}_{2,\text{ext}} + \vec{F}_{3,\text{ext}}\, dt.$$

The total momentum is

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 = M\vec{v}_{CM}.$$

These are related through

$$\vec{J} = \Delta\vec{P} = \vec{P}_f - \vec{P}_i.$$

Discussion: What is meant by the phrase "force acts at an instant"? To find the acceleration right now, all we need to know is the net force right now:

$$\vec{a} = \frac{\vec{F}_{\text{net}}}{m}.$$

But to find the velocity right now, we need to know the net forces acting back to the last time that the ball was at rest:

$$\vec{v} = \frac{\int \vec{F}_{\text{net}} dt}{m}.$$

When laypeople talk about "leftover force" (see section 4.10), they're talking about something akin to the physics term "net impulse".

2.13 Calculating the Work Done

This section finds the work done by various forces acting upon particles undergoing particular motions. The three examples are: (1) an unusual force; (2) the force of gravity near the Earth's surface; and (3) the $1/r^2$ force of gravity.

Work done by an unusual force

A particle is subject to a force of $\vec{F} = 3x\hat{\imath} + 5xy\hat{\jmath}$. (Take x and y in meters, F in newtons.) How much work is done on the particle by this force as it moves in a straight line

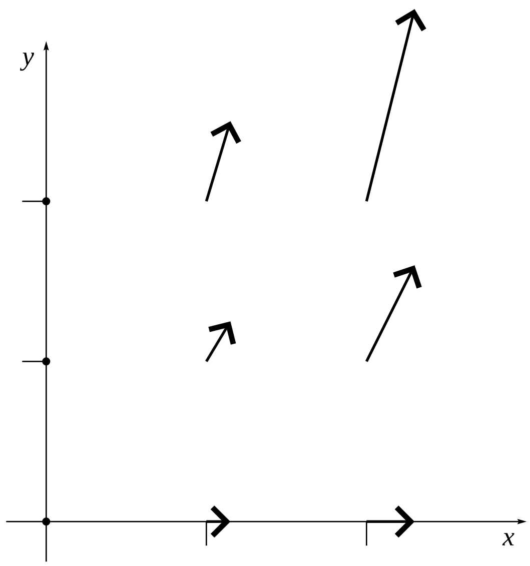
a. from $(0,0)$ to $(2,0)$?

b. from $(2,0)$ to $(2,1)$?

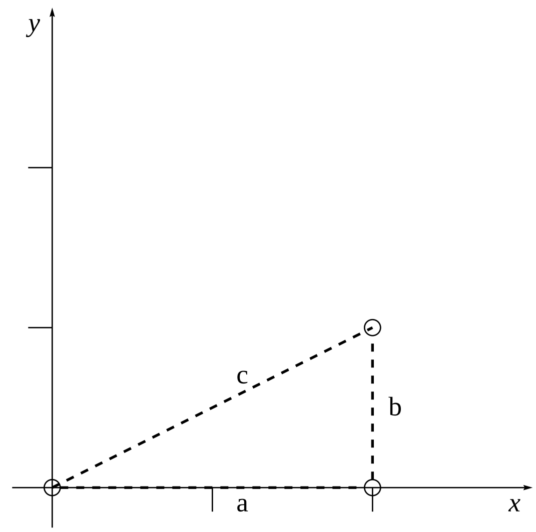
c. from $(0,0)$ to $(2,1)$?

(Of course, other forces must come into play or else the particle will not take these particular paths — I am asking about the work done by this particular force, not about the net work. Note that this force is independent of velocity, so it's irrelevant how fast the particle moves along these paths.)

The force



The paths



(The thick arrows show what the force would be if the particle were at the given position. Of course, the particle is not at all the locations at once!)

a. Path from $(0,0)$ to $(2,0)$:
On this path x goes from 0 to 2, while $y = 0$.
$\vec{F} = (3x, 5xy) = (3x, 0)$, while $d\vec{r} = (dx, 0)$, so $\vec{F} \cdot d\vec{r} = 3x\,dx$.

$$\int \vec{F} \cdot d\vec{r} = \int_0^2 3x\,dx = 3\left[\tfrac{1}{2}x^2\right]_0^2 = 6.$$

b. Path from $(2,0)$ to $(2,1)$:
On this path y goes from 0 to 1, while $x = 2$.
$\vec{F} = (3x, 5xy) = (6, 10y)$, while $d\vec{r} = (0, dy)$, so $\vec{F} \cdot d\vec{r} = 10y\,dy$.

$$\int \vec{F} \cdot d\vec{r} = \int_0^1 10y\,dy = 10\left[\tfrac{1}{2}y^2\right]_0^1 = 5.$$

c. Path from $(0,0)$ to $(2,1)$.
On this path x goes from 0 to 2, while $y = \frac{1}{2}x$.
$\vec{F} = (3x, 5xy) = (3x, \frac{5}{2}x^2)$, while $d\vec{r} = (dx, dy) = (dx, \frac{1}{2}dx)$, so $\vec{F} \cdot d\vec{r} = 3x\,dx + \frac{5}{4}x^2\,dx = (3x + \frac{5}{4}x^2)dx$.

$$\int \vec{F} \cdot d\vec{r} = \int_0^2 (3x + \tfrac{5}{4}x^2)dx = \left[3(\tfrac{1}{2}x^2) + \tfrac{5}{4}(\tfrac{1}{3}x^3)\right]_0^2 = 6 + \tfrac{10}{3} = 9\tfrac{1}{3}.$$

Moral of the story: The work done depends on the path! Walking directly from $(0, 0)$ to $(2, 1)$ results in a work of $9\frac{1}{3}$ joules. Walking from $(0, 0)$ to $(2, 1)$ via $(2, 0)$ results in a work of 11 joules.

Work done by gravity (near the Earth's surface)

A particle near the earth's surface is subject to the gravitational force of $\vec{F} = 0\,\hat{\imath} - mg\,\hat{\jmath}$. How much work is done on the particle by gravity as it moves from $(0, 0)$ to $(7, 3)$ following each of the following paths? (All distances are in meters.)

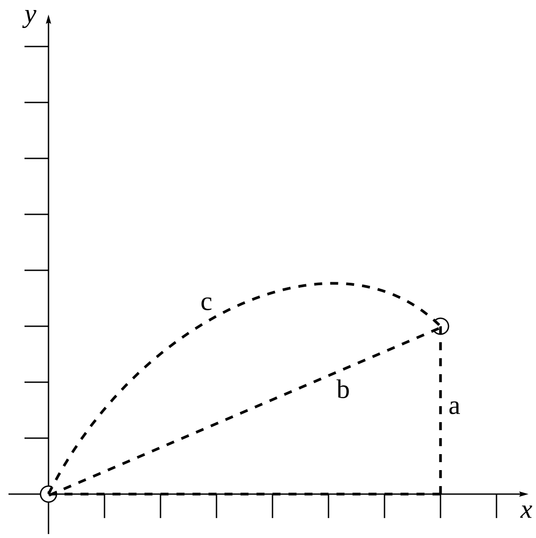
a. Horizontally from $(0, 0)$ to $(7, 0)$, then vertically to $(7, 3)$.

b. In a straight line.

c. Along the parabola $y = -\frac{11}{49}x^2 + 2x$.

(Note that we didn't have to place the origin at the initial point: We could have talked about moving from $(1, 2)$ to $(8, 5)$ and gotten the same answer, but finding that answer would have involved considerably more labor. Similarly, we didn't need to take $\hat{\imath}$ horizontal and $\hat{\jmath}$ vertical: we could have tilted the axes by 17°. You can do all sorts of things to make life hard on yourself!)

The paths



a. Part I: Horizontally from $(0, 0)$ to $(7, 0)$:
On this path x goes from 0 to 7, while $y = 0$.
$\vec{r} = (x, 0)$, so $d\vec{r} = (dx, 0)$.
$\vec{F} = (0, -mg)$, so $\vec{F} \cdot d\vec{r} = 0\, dx$.

$$\int \vec{F} \cdot d\vec{r} = \int_0^7 0\, dx = 0.$$

This makes sense: On a horizontal path the displacement and force of gravity are perpendicular, so their dot product is zero and gravity does no work.

Part II: Vertically from $(7, 0)$ to $(7, 3)$:
On this path y goes from 0 to 3, while $x = 7$.
$\vec{r} = (7, y)$, so $d\vec{r} = (0, dy)$.
$\vec{F} = (0, -mg)$, so $\vec{F} \cdot d\vec{r} = -mg\, dy$.

$$\int \vec{F} \cdot d\vec{r} = \int_0^3 -mg\, dy = -mg(3).$$

So the total work done by gravity on the particle taking this path is $0 - mg(3) = -mg(3)$.

b. A straight line from $(0, 0)$ to $(7, 3)$:
On this path x goes from 0 to 7, while $y = \frac{3}{7}x$.
$\vec{r} = (x, \frac{3}{7}x)$, so $d\vec{r} = (dx, \frac{3}{7}\, dx)$.
$\vec{F} = (0, -mg)$, so $\vec{F} \cdot d\vec{r} = -\frac{3}{7}mg\, dx$.

$$\int \vec{F} \cdot d\vec{r} = \int_0^7 -\tfrac{3}{7}mg\,dx = -\tfrac{3}{7}mg[x]_0^7 = -mg(3).$$

The total work done by gravity is again $-mg(3)$.

c. The parabola $y = -\frac{11}{49}x^2 + 2x$ from $(0,0)$ to $(7,3)$:
(First of all, you should verify that this parabola does in fact pass through the origin and through $(7,3)$.)
On this path x goes from 0 to 7, while $y = -\frac{11}{49}x^2 + 2x$.
$\vec{r} = (x, -\frac{11}{49}x^2 + 2x)$, so $d\vec{r} = (dx, [-\frac{22}{49}x + 2]\,dx)$.
$\vec{F} = (0, -mg)$, so $\vec{F} \cdot d\vec{r} = -mg[-\frac{22}{49}x + 2]\,dx$.

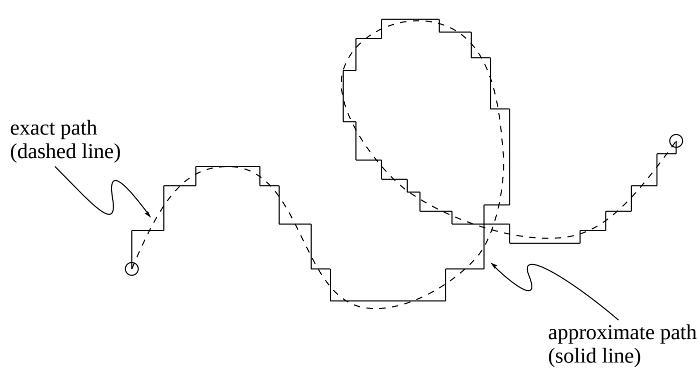
$$\int \vec{F} \cdot d\vec{r} = \int_0^7 -mg[-\tfrac{22}{49}x + 2]\,dx = -mg[-\tfrac{22}{49} \cdot \tfrac{1}{2}x^2 + 2x]_0^7 = -mg[-\tfrac{22}{49} \cdot \tfrac{1}{2}7^2 + 2 \cdot 7] = -mg(3).$$

The total work done by gravity is yet again $-mg(3)$.

Moral of the story: We know from our previous calculation that the work done in moving from one point to another might depend on the path taken. But these three examples suggest that if the force happens to be gravity, then the work happens to be the same for each path! You can prove this to yourself by approximating any arbitrary path (even one with loop-de-loops) by a staircase with horizontal treads and vertical risers. (We can make this approximation as accurate as we please by considering a very large number of very short treads and risers. In a math class, this would be called "taking a limit".) Gravity does no work along the horizontal treads. The total work done when adding the work due to each vertical riser is just

$$-mg\,(\text{increase in height}).$$

Thus the work done by gravity is not only independent of path (i.e. depends only on endpoints), it's even given by a simple formula.



Work done by gravity (possibly far from the Earth's surface)

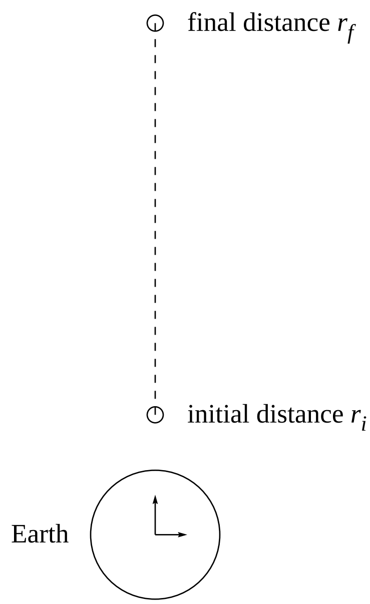
A particle of mass m is subject to the Earth's gravitational force with magnitude

$$F = G\frac{mM_E}{r^2}$$

and direction towards the center of the Earth. (Here M_E is the mass of the Earth and r is the distance to the center of the Earth.)

Moving straight up

How much work is done on the particle by the gravitational force if the particle starts a distance r_i from the center of the Earth and moves straight up to a distance r_f from the center of the Earth?



With the coordinate system shown, we have

$$\vec{F} = -G\frac{mM_E}{r^2}\,\hat{\jmath}$$

and

$$\vec{r} = r\,\hat{\jmath}.$$

Thus

$$\vec{F}\cdot d\vec{r} = -G\frac{mM_E}{r^2}\,dr$$

and

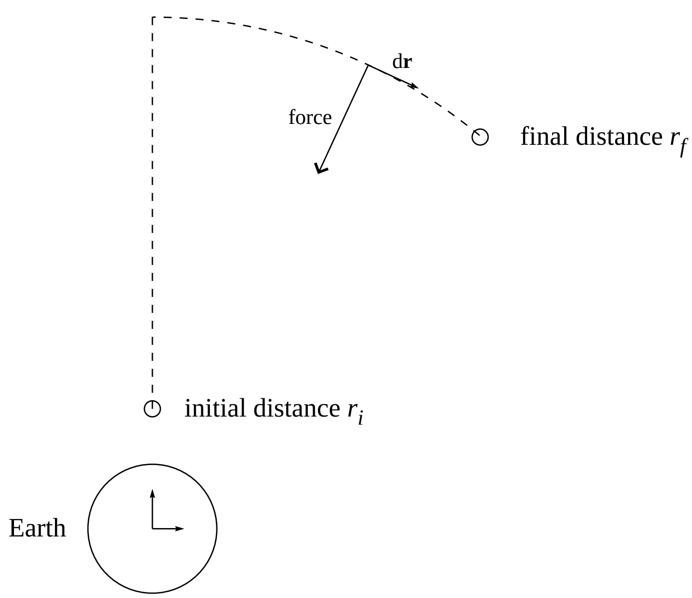
$$\begin{aligned}\int \vec{F}\cdot d\vec{r} &= \int_{r_i}^{r_f}\left[-G\frac{mM_E}{r^2}\right]dr \\ &= -GmM_E\int_{r_i}^{r_f}\frac{1}{r^2}\,dr\end{aligned}$$

$$
\begin{aligned}
&= -GmM_E\left[-\frac{1}{r}\right]_{r_i}^{r_f} \\
&= +GmM_E\left[\frac{1}{r}\right]_{r_i}^{r_f} \\
&= \frac{GmM_E}{r_f} - \frac{GmM_E}{r_i}.
\end{aligned}
$$

If $r_f > r_i$, as in the figure, the work done by gravity is negative.

Moving up then over

How much work is done on the particle by the gravitational force if the particle starts and ends at the same distances as before, but instead of moving straight up, if moves first up and then over?

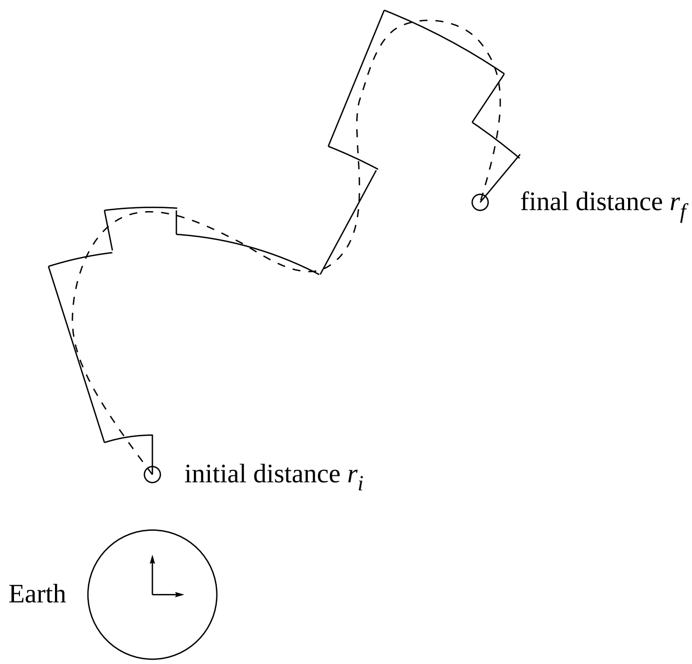


The total work is the sum of the work done moving straight up plus the work done moving over along the arc. While the particle is moving along the arc, the force due to gravity is purely radial, while the infinitesimal displacement $d\vec{r}$ is purely tangential. Thus they are perpendicular and $\vec{F}\cdot d\vec{r} = 0$. Gravity does no work while the particle moves along the arc. The work done by gravity when the particle takes this path is the same as it was last time, namely

$$\int \vec{F}\cdot d\vec{r} = \frac{GmM_E}{r_f} - \frac{GmM_E}{r_i}.$$

Moving in a squiggle

What if the path from initial to final is the squiggle shown below? In this case we approximate the squiggle by a series of rays and arcs. (As always, we can make this approximation more and more accurate by using more and more rays and arcs, each of which will be shorter and shorter.)



It's clear that gravity does no work on the arcs. And a moment's reflection shows that the work done on the rays is the same as it was when we just went up and then over. (The work done going out beyond r_f is the negative of the work done coming back in again.) The work done by gravity is independent of the path and depends only on the initial and final distances, and is given by

$$\int \vec{F} \cdot d\vec{r} = \frac{GmM_E}{r_f} - \frac{GmM_E}{r_i}.$$

The "near the Earth's surface" limit

Does this general expression have the proper limit for motions near the surface of the Earth? It doesn't look as if it could, because the general expression involves G while the expression valid for motions near the Earth's surface involves g. Let's not give up yet, however.

In general, the gravitational force on a body of mass m due to the Earth of mass M_E is

$$G\frac{mM_E}{r^2}.$$

Near the Earth's surface (i.e. $r = R_E$), the gravitational force is

$$mg.$$

Equating these two expressions tells us that

$$g = \frac{GM_E}{R_E^2}.$$

How does this relate to our expressions for work? Near the Earth's surface, the work done by gravity when an body increases in height by h is

$$-mgh.$$

But the general expression for the work done by gravity is

$$\frac{GmM_E}{r_f} - \frac{GmM_E}{r_i}.$$

If $r_i = R_E$ and $r_f = R_E + h$, then this general expression becomes

$$GmM_E\left(\frac{1}{R_E + h} - \frac{1}{R_E}\right) = GmM_E\left(\frac{1}{R_E(1 + h/R_E)} - \frac{1}{R_E}\right).$$

Now use the approximation,[8] valid for $|\epsilon| \ll 1$, that

$$\frac{1}{1+\epsilon} \approx 1 - \epsilon.$$

In our case $h/R_E \ll 1$, so the work done by gravity is approximately

$$GmM_E\left(\frac{1}{R_E(1 + h/R_E)} - \frac{1}{R_E}\right) \approx GmM_E\left(\frac{1}{R_E}\right)\left(1 - \frac{h}{R_E} - 1\right) = -m\frac{GM_E}{R_E^2}h.$$

Using the expression for g above, this becomes

$$-mgh$$

as required!

Connection to multivariate calculus

According to the definition of work, the work done when a particle moves from $\vec{r}_i$ to $\vec{r}_f$ will generally depend not only on the endpoints but also on the path taken from one endpoint to the other.

[8]This is a truncation of the "geometric series"

$$\frac{1}{1+\epsilon} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \epsilon^4 - \cdots$$

which converges when $|\epsilon| < 1$.

But we've seen here that in several important cases, the work done happens to be independent of path, and depends only upon the endpoints! Those of you taking multivariate calculus will find the following result (which I quote without proof) interesting.

Suppose there is a vector function

$$\vec{F}(\vec{r}) = (F_x(x,y,z), F_y(x,y,z), F_z(x,y,z))$$

such that $\vec{F}(\vec{r})$ is the force that would act on a particle if it were placed at $\vec{r}$. Then the work done by this force will be independent of path if and only if

$$\operatorname{curl} \vec{F}(\vec{r}) = \vec{0}.$$

Furthermore, if this is the case then there exists a scalar function $u(\vec{r})$ such that

$$\vec{F}(\vec{r}) = -\operatorname{grad} u(\vec{r}).$$

The function $u(\vec{r})$ is the potential energy that the system would have if the particle were placed at $\vec{r}$.

2.14 Momentum versus Kinetic Energy

Physics is becoming more abstract:

Position and speed were familiar from everyday life.
Acceleration is a little more iffy.
Force seems more familiar, although Newton's third law makes clear that physics "force" is not the same as everyday "force."
But momentum and kinetic energy??

Both of these are measures of the "oomph" of motion. How do they differ?

The quantity "momentum", $m\vec{v}$, was introduced by Newton, who called it the "quantity of motion."
The quantity "kinetic energy", $\frac{1}{2}mv^2$, was introduced by Leibniz, who called it the "liveliness" (or "vitality") of motion.

But to get a real idea of the distinction, consider the following one-dimensional problem: *Suppose a bullet is shot into a sand bank where it experiences a constant retarding force $-F$ until it comes to rest.* Then:

$$\begin{aligned} \Delta p = p_{\text{final}} - p_{\text{initial}} &= (-F)\Delta t \quad \text{so} \quad p_{\text{initial}} = F\Delta t \\ \Delta \text{KE} = \text{KE}_{\text{final}} - \text{KE}_{\text{initial}} &= (-F)\Delta x \quad \text{so} \quad \text{KE}_{\text{initial}} = F\Delta x. \end{aligned}$$

Suppose I shoot one bullet, then a second bullet at twice the speed. The second bullet has twice the momentum, so it takes twice the time to stop. The second bullet has four times the kinetic energy, so it takes four times the distance to stop. Thus the amount of sand that the bullet shoves aside is proportional to the kinetic energy.

Similarly, a ball thrown into the window of a china shop with twice the speed will do four times the damage before coming to rest. (It will take twice as much time to do that damage, but the shop owner is probably more concerned with the amount of damage done than with the time required to do the damage.)

When my son Colin was three years old and he visited his friends, the mothers would say afterwards "He has a lot of energy today" meaning "He broke a lot of toys" or "He made a real mess of things".

This might give you the impression that kinetic energy is intrinsically destructive. (Or that Colin is intrinsically destructive!) This is not so — I'll give examples later of "constructive" effects of kinetic energy. In general, kinetic energy is a measure of how much "crashing and smashing" an object *can do* before coming to rest, not how much it *has to* do.

(Paul Hewitt's *Conceptual Physics* provides further insight into the distinction between momentum and kinetic energy.)

Yes: momentum and energy *are* abstract quantities: you can't see them, or taste them, or put them in a box. But just because an idea's abstract doesn't mean it's unimportant. Here are some other abstract ideas: the number 2, the concept time, love, freedom, liberty, fraternity, equality. People pay money for abstract ideas — people fight and die for them. Calling an idea abstract is no insult.

2.15 Work and Energy

- *Work.* A single particle subject to several forces moves along path $\vec{r}(t)$ from point $\vec{r}_i$ to point $\vec{r}_f$. The work done by any particular force $\vec{F}_P$ — a force that might vary depending on the particle's position, or velocity, or even the wind speed experienced by the particle — is

$$W_P = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_P \cdot d\vec{r}.$$

  This work will, in general, depend on the precise path taken by the particle from $\vec{r}_i$ to $\vec{r}_f$.

- *Work-Kinetic Energy Theorem.* The net work done on a particle equals the change in its kinetic energy. For example: If there are three forces acting on the particle — call them L, M, and N — then

$$W_L + W_M + W_N = K_f - K_i.$$

  This is a remarkable and unexpected result because each work term will, in general, depend upon path, whereas the change in kinetic energy depends only upon the situation at the end points.

- *Potential Energy.* But there are even more remarkable things in store. For some special forces — such as gravity, and the spring force — the work done depends not on path but just the end points. Such forces are called "conservative":

$$W_C = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_C \cdot d\vec{r} = -[U_C(\vec{r}_f) - U_C(\vec{r}_i)].$$

- *Conservation of Energy.* Suppose that of the three forces L, M, and N acting on our particle, forces L and M are conservative while force N is not. Then the work-kinetic energy theorem becomes

$$W_N = [K_f + U_L(\vec{r}_f) + U_M(\vec{r}_f)] - [K_i + U_L(\vec{r}_i) + U_M(\vec{r}_i)].$$

  In words, the work due to non-conservative forces equals the change in energy of the particle. If all the forces are conservative, then energy is conserved.

For a system of particles:

It is *not* true that $K = \frac{1}{2}Mv_{CM}^2$. (For example, if the center of mass is stationary and the system spins about the CM, then the spinning contributes to the kinetic energy.)

It is *not* true that Work due to external forces $= \Delta K$. (Whereas the impulse due to internal forces sums to zero, the work due to internal forces might not.)

2.16 The Word "Energy"

You are by now familiar with the story that "energy" means one thing in everyday discourse ("the play was performed with enormous dramatic energy") and another thing in physics. What is new about the word "energy" is that it has two meanings in scientific discourse.

The first scientific meaning is energy as discussed in this course. This is the proper use of the term.

But in connection with thermal processes the term "energy" is often (and improperly) used to mean what should be called "Gibbs potential." When a biologist says "the compound adenosine triphosphate (ATP) is a high-energy molecule" or "energy is released in the reaction

$$\mathrm{ATP}^{3-} + \mathrm{H_2O} \to \mathrm{ADT}^{2-} + \mathrm{H_2PO_4^-}\text{"}$$

or "life on earth is dependent on energy from the sun" the biologist really means, not energy, but Gibbs potential. You will learn more about Gibbs potential in Physics 111. For now, I'll just say that Gibbs potential takes into account both energy and entropy effects.

You have been told to "turn off the lights to conserve energy." This makes no sense — energy is conserved whether the lights are on or off. The expression should be "turn off the lights to conserve Gibbs potential."

To make matters still more confusing, Gibbs potential is also called "Gibbs free energy." This is a real misnomer, because Gibbs potential is rarely free (in the economic sense of the word "free"). Indeed, when you pay your energy bill (to a "power company" which should be called a "Gibbs potential company") you are paying, not for energy, but for Gibbs potential.

I am trying to get the federal government to change the "Department of Energy" into the "Department of Gibbs Potential," but so far I've made no progress.

2.17 Particles vs. Systems

A system consists of N particles, each of mass m_i, for a total mass of

$$M = \sum_{i=1}^{N} m_i.$$

The center of mass is located at

$$\vec{r}_{CM} = \frac{\sum\limits_{i=0}^{N} m_i \vec{r}_i}{M}.$$

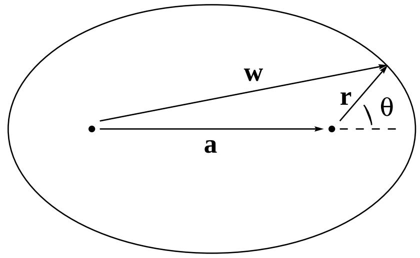
For a particle:	**For a system:**
$\vec{a} = \frac{\sum \vec{F}}{m}$	$\vec{a}_{CM} = \frac{\sum \vec{F}_{\text{ext}}}{m}$
$\vec{p} = m\vec{v}$	$\vec{P} = \sum_{i=1}^{N} m_i \vec{v}_i = M\vec{v}_{CM}$
$\sum \vec{F} = \frac{d\vec{p}}{dt}$	$\sum \vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt}$
$K = \frac{1}{2}mv^2$	$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 \neq \frac{1}{2} M v_{CM}^2$
$\sum \text{work} = \Delta K$	$\sum \text{external work} \neq \Delta K$

2.18 Elliptical Orbits

You have undoubtedly heard that planets travel in elliptical orbits about the sun — Kepler discovered this fact empirically on about Easter of 1605. We show here that this fact follows from Newtonian gravitational attraction.

Ellipses (geometry)

In order to show that the equations of physics give rise to an ellipse, we must first use geometry find the equation for an ellipse.



The definition ("pins and strings") of an ellipse starts with two points called foci (singular: focus). The ellipse consists of those points for which the distance to one focus, plus the distance to the other focus, is some constant. In the figure above these two foci are separated by distance a, and the distances are r and w, so each point on the ellipse has

$$w + r = \text{ constant } = a/\epsilon.$$

There is no conventional name for this constant. Instead, it is conventional to define the "eccentricity"

$$\epsilon = \frac{a}{w + r}.$$

For a circle, $a = 0$ so the eccentricity is $\epsilon = 0$. The more an ellipse deviates from circularity, the greater its eccentricity.

From this relation, we seek an equation for $r(\theta)$. Begin with

$$\vec{w} = \vec{r} + \vec{a}$$

and dot each side with itself to find

$$\vec{w} \cdot \vec{w} = (\vec{r} + \vec{a}) \cdot (\vec{r} + \vec{a}) = \vec{r} \cdot \vec{r} + 2\vec{r} \cdot \vec{a} + \vec{a} \cdot \vec{a}$$

or

$$w^2 = r^2 + 2ra\cos\theta + a^2. \tag{2.36}$$

This is indeed an equation for $r(\theta)$, but it also involves $w(\theta)$, and we need to eliminate that quantity.

To do so, take the derivative of both sides with respect to θ:

$$2w\frac{dw}{d\theta} = 2r\frac{dr}{d\theta} + 2\frac{dr}{d\theta}a\cos\theta - 2ra\sin\theta.$$

But $w + r = $ constant, so

$$\frac{dw}{d\theta} = -\frac{dr}{d\theta}$$

and

$$\begin{aligned}
-w\frac{dr}{d\theta} &= r\frac{dr}{d\theta} + \frac{dr}{d\theta}a\cos\theta - ra\sin\theta \\
0 &= (w + r + a\cos\theta)\frac{dr}{d\theta} - ra\sin\theta \\
0 &= (a/\epsilon + a\cos\theta)\frac{dr}{d\theta} - ra\sin\theta \\
0 &= (1/\epsilon + \cos\theta)\frac{dr}{d\theta} - r\sin\theta
\end{aligned}$$

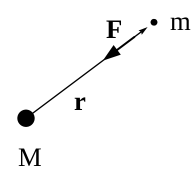
or, finally,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\epsilon \sin\theta}{1 + \epsilon\cos\theta}. \tag{2.37}$$

This is the differential equation satisfied by an ellipse.

Orbits (physics)

A planet of mass m orbits a star of mass M. (The star is so much more massive than the planet that the star may be considered fixed.) What does physics say about planet's motion?



We could start by applying $\vec{F} = m\vec{a}$ to the planet, but it's much more efficient to use the conservation laws. Let's go through them in sequence: (1) The planet's momentum is not conserved because there is a (gravitational) force on the planet. (2) The gravitational energy of the sun-planet system *is* conserved, because gravity is a conservative force. In fact our discussion of work in gravity demonstrated that the total energy

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} \tag{2.38}$$

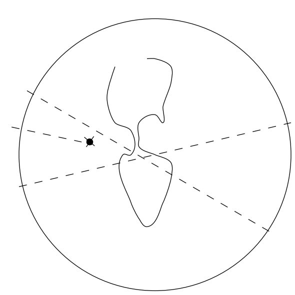
is conserved. (3) The planet's angular momentum about the star *is* conserved, because the gravitational force is parallel to $\vec{r}$ so the torque applied is zero. Thus

$$\vec{\ell} = m\vec{r} \times \vec{v} \tag{2.39}$$

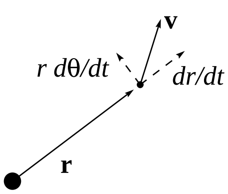
is conserved.

The vector character of angular momentum tells us that the planet always moves within a single plane — i.e. that $\vec{r}$ and $\vec{v}$ are always within the plane perpendicular to $\vec{\ell}$. Occasionally one sees animations of satellite motion in which the satellite changes the plane of its orbit. . . you can be confident that such motions are the product of the animator's mind and that real satellites don't behave this way.

NO!



How do the conservation laws look in terms of functions $r(t)$ and $\theta(t)$? This figure shows that the radial component of the velocity is dr/dt, while the tangential component is $r\,d\theta/dt$.



Thus

$$E = \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{GMm}{r} \tag{2.40}$$

while

$$\ell = mr^2\frac{d\theta}{dt}. \tag{2.41}$$

These are two differential equations for the two unknown functions $r(t)$ and $\theta(t)$. We *could* solve the two equations, but it would be hard, and it really wouldn't give us what we wanted: We want $r(\theta)$.

To find an equation for $r(\theta)$ we use the chain rule,

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt},$$

followed by the conservation of momentum

$$\frac{d\theta}{dt} = \frac{\ell}{mr^2},$$

to turn the energy conservation equation into

$$E = \frac{1}{2}m\left[\left(\frac{dr}{d\theta}\frac{\ell}{mr^2}\right)^2 + \left(\frac{\ell}{mr}\right)^2\right] - \frac{GMm}{r}.$$

Minor rearrangement gives

$$\begin{aligned} E &= \frac{1}{2}m\frac{\ell^2}{m^2r^2}\left[\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 + 1\right] - \frac{GMm}{r} \\ E + \frac{GMm}{r} &= \frac{\ell^2}{2mr^2}\left[\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 + 1\right] \\ \left[E + \frac{GMm}{r}\right]\frac{2mr^2}{\ell^2} &= \left[\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 + 1\right] \end{aligned}$$

or, finally,

$$\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{\ell^2}r^2 + \frac{2GMm^2}{\ell^2}r - 1. \tag{2.42}$$

Now we're at a point where we can compare our physics and geometry results! Geometry tells us that $r(\theta)$ for an ellipse is given by equation (2.37). Physics tells us that $r(\theta)$ for a planet is given by equation (2.42). These two equations seem to have nothing in common! Did we really screw up that badly?

No. The geometry tells us $dr/d\theta$ in terms of θ (measured with respect to the semimajor axis) whereas physics tells us $dr/d\theta$ in terms of r — and the physics scheme never singles out one axis as being special. We could convert the physics scheme into the geometry one, but it's more convenient to convert the geometry scheme into the physics one.

Ellipses (geometry) again

We need to find an expression for $\cos\theta$ in terms of r, then use $\sin^2\theta + \cos^2\theta = 1$ to find $\sin\theta$ in terms of r, and then plug these two results into equation (2.37) to find a geometrical result for $dr/d\theta$ in terms of r. Starting with equation (2.36),

$$\begin{aligned}
\cos\theta &= \frac{w^2 - r^2 - a^2}{2ra} \\
&= \frac{(a/\epsilon - r)^2 - r^2 - a^2}{2ra} \\
&= \frac{a^2/\epsilon^2 - 2ar/\epsilon - a^2}{2ra} \\
&= \frac{a^2(1/\epsilon^2 - 1) - 2ar/\epsilon}{2ra} \\
&= \frac{a(1/\epsilon^2 - 1)}{2r} - \frac{1}{\epsilon}.
\end{aligned}$$

Thus

$$\cos^2\theta = \frac{a^2(1/\epsilon^2 - 1)^2}{4r^2} - \frac{a(1/\epsilon^2 - 1)}{r\epsilon} + \frac{1}{\epsilon^2}$$

and

$$\sin^2\theta = 1 - \cos^2\theta = -\frac{a^2(1/\epsilon^2 - 1)^2}{4r^2} + \frac{a(1/\epsilon^2 - 1)}{r\epsilon} + 1 - \frac{1}{\epsilon^2}.$$

If we square equation (2.37) and then plug in the above expressions for $\cos\theta$ and $\sin^2\theta$, we find

$$\begin{aligned}
\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 \left(\frac{1}{\epsilon} + \cos\theta\right)^2 &= \sin^2\theta \\
\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 \left(\frac{a(1/\epsilon^2 - 1)}{2r}\right)^2 &= -\frac{a^2(1/\epsilon^2 - 1)^2}{4r^2} + \frac{a(1/\epsilon^2 - 1)}{r\epsilon} + 1 - \frac{1}{\epsilon^2} \\
\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 &= -1 + \frac{4r}{a\epsilon(1/\epsilon^2 - 1)} - \frac{4r^2}{a^2(1/\epsilon^2 - 1)}.
\end{aligned}$$

Geometry compared with physics

Comparing the geometry result

$$\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 = -\frac{4}{a^2(1/\epsilon^2 - 1)}r^2 + \frac{4}{a\epsilon(1/\epsilon^2 - 1)}r - 1$$

with the physics result (equation 2.42)

$$\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{\ell^2}r^2 + \frac{2GMm^2}{\ell^2}r - 1$$

shows that these two equations are exactly the same, provided that the two physics parameters (E and ℓ) are related to the two geometry parameters (a and ϵ) through

$$\begin{aligned} E &= -\frac{GMm}{a}\epsilon \\ \ell^2 &= \tfrac{1}{2}GMm^2a\frac{1-\epsilon^2}{\epsilon}. \end{aligned}$$

Thus a planet moves in an elliptical orbit with the sun located at one focus.

Chapter 3

Additional Problems in Classical Mechanics

The letters HRW refer to the text by D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*, seventh edition (John Wiley, New York, 2005).

1. *Weighting light things with a heavy standard* (HRW section **1**–2)

You are given an equal-arm balance, a replica of the standard kilogram, a single apple (which is much less massive than a kilogram), and a big blob of clay. How can you accurately find the apple's mass?

2. *Significant figures* (HRW section **1**–4)

Perform the following calculations and express the answers to the correct number of significant figures: (a) Multiply 3.4 by 7.954. (b) Add 99.3 and 98.7. (c) Subtract 98.7 from 99.3. (d) Evaluate the cosine of 3.2°. (e) Five planks have an average length of 2.134 meters. What is the length of these five planks laid end-to-end?

3. *A new law of nature?* (HRW section **1**–4)

It has been proposed that the speed of sound v_s and the speed of light c are related through $v_s = \frac{1}{2}\sqrt[3]{c}$. Check the accuracy of this formula using speeds expressed in meters/second, then recheck its accuracy using speeds expressed in kilometers/second. (According to the book *U.S. Standard Atmosphere, 1976*, the standard speed of sound at sea level is 340.29 m/s.) Is this proposal a new and surprising law of nature, or merely a coincidence? Explain.

4. *Physics in film* (HRW section **1**–4)

Alfred Hitchcock's 1935 film *The 39 Steps* is one of the great spy thrillers of all time. In the film, several men and women travel across England and Scotland in pursuit of an important but unspecified secret document. Only in the final minute of the film does the audience find that the the document contains the specifications for a completely silent aircraft engine, and that these specifications hinge upon "the secret formula

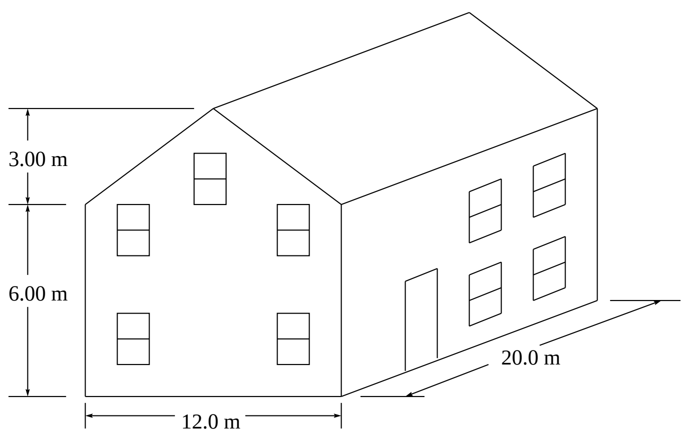
$$\left(r - \frac{1}{r}\right)^{\gamma}$$

where r represents the ratio of compression and γ the axis of the fluid line of the cylinder."

Show that this formula is not worth the pursuit of dozens of spies, and in fact is entirely without meaning.

Clue: You know that 5^3 means "5 times itself three times" or $5^3 = 5 \times 5 \times 5$. What does $5^{(3 \text{ feet})}$ mean? How is it related to $5^{(1 \text{ yard})}$?

5. *A Doll's House* (HRW section **1**–5)
In the United States, a doll house has the scale of 1:12 of a real house. That is, each length of a doll house is $\frac{1}{12}$ the corresponding length of a real house. Similarly, a miniature house (a doll house to fit within a doll house) has the scale of 1:12 of a doll house or 1:144 of a real house. Suppose the real house has the dimensions shown in the sketch. Find the volume, in cubic meters, of the corresponding (a) doll house and (b) miniature house. (Warm up question: If the length of every edge of a cube is doubled, what happens to the volume of the cube?)



6. *Model train* (HRW section **1**–5)
The Clinch Park Zoo in Traverse City, Michigan, maintains a working, quarter-scale model of a "4-4-2 (Atlantic)" steam locomotive.[1] By "quarter-scale" I mean that the model is one-forth as long, one-fourth as wide, and one-fourth as tall as a regular 4-4-2 (Atlantic) steam locomotive, such as the Southern-Pacific's locomotive #3025, which is now at the Travel Town Museum in Los Angeles. The #3025 locomotive weights 115 tons. Estimate the weight of the model locomotive.

7. *Map download times*
The website http://www.topozone.com contains topographic maps of the entire United States. After you

[1] A photograph is available at http://photos.igougo.com/pictures-photos-p69670-Clinch_Park_Train.html.

enter a location, topozone provides a rectangle of topographic map centered on that location. You can select either a "small view" (about 4 by 5 inches) or a "large view" (about 2.5 times larger on each side). If a "small view" map takes four seconds to download, about how long will it take a "large view" map to download?

8. *Apples*
Which has more apple skin: A pound of large apples or a pound of small apples? A bushel of large apples or a bushel of small apples?

9. *Giants in the North Pacific* (HRW section **1**–5)
An expedition to a remote island in the North Pacific is surprised to discover a race of giant humans. These so-called "Islanders" are made of the same organs and tissues that other people are — the same sort of lungs, the same kind of bone tissue, and so forth — but a typical Islander is twice as tall, twice as wide, and twice as thick as a typical American.

a. How much does a typical Islander weigh relative to a typical American?

The weight of an Islander is of course supported by by his or her leg bone. The weight-supporting capacity of a leg bone (or any other column) is proportional to the cross-sectional area of that bone.

b. How wide is a typical Islander leg bone relative to a typical American leg bone?

10. *Birth and growth*
Kira Ijiri FitzGerald was born on 4 September 2002 with a height of 20.0 inches and weight of 7.2 pounds. At her four-month checkup, she was found to have grown to 25.5 inches.

a. Estimate Kira's weight at her four-month checkup.

Soon after her birth, Kira wore a one-piece "snugli" outfit that weighed 6.2 ounces. She outgrew that snugli and went to her four-month checkup dressed in a snugli of the same style made out of fabric of the same thickness.

b. Estimate the weight of the snugli Kira wore to her four-month checkup.

11. *Watersheds*
The Black River (just east of Oberlin) and the Vermilion River (just west of Oberlin) drain watersheds of similar geological and meteorological character. The Vermilion River watershed has an area of 260 square miles, and it discharges about 21 cubic feet of water each second into Lake Erie. The Black River watershed has an area of 520 square miles.

a. Estimate the discharge rate of the Black River. Explain your estimate briefly but cogently.

The Vermilion River, plus all its tributaries, has a total length of 41 miles.

b. Do you expect the total length of the Black River, plus all its tributaries, to be about 82 miles, or more, or less? Explain your answer briefly but cogently.

12. *Mob fights*

A mob fight normally starts when two people begin scuffling, and then the fracas spreads.

a. Each pair of people is a potential scuffle. In a group of N people, how many pairs of people are there? Check your formula for the special cases $N = 1$, $N = 2$, and $N = 3$.

b. It is found that a crowd of 1,000 people requires a security force of 30 officers to maintain control. How many officers will be needed for a crowd of 2,000 people?

13. *Size of a cell*

Model a biological cell as a sphere of radius r. The cell has nutrient needs proportional to its volume ($\propto r^3$) and ability to absorb nutrients proportional to its surface area ($\propto r^2$). A larger cell also has the ability to do more things. This "capability" measure increases with r faster than volume does, but it's not clear exactly how much faster — let's assume it is proportional to r^4. Thus in this model the "quality measure" of a cell is

$$Ar^4 - Br^3 + Cr^2$$

where A, B, and C are positive constants. Find a formula for the radius of the cell of highest quality in terms of A, B, and C, and find also a restriction on the constants for this formula to make sense.

14. *Three jolly fishermen*

Three fishermen paddle out to an island, each in his own canoe. The friends set up camp, have dinner, and spend the evening catching a big pile of fish. The sun sets and they go to sleep.

One fisherman has difficulty falling to sleep and decides to paddle back home. Without waking his snoring companions, he decides that one-third of the fish are rightfully his. He divides the fish into three equal piles, finds that there's one fish left over, throws that last fish back into the lake, takes one of the three equal piles, and leaves in his own canoe.

Another fisherman wakes up confused in the night, and decides to paddle back home. Unaware that one friend has already departed, he decides that one-third of the fish are rightfully his. He divides the fish into three equal piles, finds that there's one fish left over, throws that last fish back into the lake, takes one of the three equal piles, and leaves in his own canoe.

The third fisherman wakes up still later in the night, and decides to paddle back home. Unaware that he is alone on the island, he decides that one-third of the fish are rightfully his. He divides the fish into three equal piles, finds that there's one fish left over, throws that last fish back into the lake, takes one of the three equal piles, and leaves in his own canoe.

What is the smallest possible catch for that evening?

15. *How long is a lecture?* (HRW section **1**–6)
Physicist Enrico Fermi found an amusing result when he decided to express the time of a typical class meeting — 50 minutes — in terms of microcenturies. (a) What was it? (Take the time of a lecture to have two significant figures, since they often run over and occasionally run short.) (b) What is the percentage difference resulting from approximating the class meeting time as 1.00 microcentury? Use

$$\text{percentage difference} = \left(\frac{\text{actual} - \text{approximate}}{\text{actual}}\right) \times 100\%.$$

16. *To see a world in a grain of sand...* (HRW section **1**–7)
The grains of sand on a beach vary in size and shape, but on one Lake Erie beach the grains are approximate spheres with an average radius of about 50 μm. Sand is mostly silicon dioxide (quartz), and a solid block of $Si0_2$ with a volume of 1.00 m^3 has a mass of 2600 kg. What mass of sand grains would provide a total surface area (i.e. total area of all the individual grains) equal to that of a cube 1 meter on an edge?

17. *Threads and sheets*
A spool of thin cotton thread has 12 grams of thread, which is 1,500 m long.

a. What is the linear mass density of cotton thread? That is, assuming the thread is uniform, what is the mass of one meter of thread? (The frequently-encountered "volume density" ρ has the dimensions grams/meter^3, but "linear density" λ has the dimensions grams/meter.)

b. Land's End advertises that its cotton sheets have "85 threads per centimeter". What is the surface mass density of a Land's End sheet? (Surface density σ has the dimensions grams/meter^2.)

18. *Marathon training*
Here are three rules of thumb[2] concerning training for long-distance running races such as the marathon (26 miles) or the half-marathon (13 miles):

- You need to run training mileage each week.
- Your longest race should be 1/3 your weekly training mileage.
- You should increase your weekly training mileage by at most two miles each week.

Suppose a non-runner starts training for a race adhering to these guidelines.

a. Show that it takes twice as many weeks to train for a marathon as it does for a half-marathon.

b. Show that the runner puts in four times the training mileage while preparing for a marathon relative to a half-marathon. (*Clue:* You can work out numbers, or you can work out equations, but if you plot weekly training mileage versus weeks in training, you'll see a simple geometric argument proving this result.)

[2] Bob Glover, *The Runner's Handbook.*

c. Compare preparation for a "sesquimarathon" (39 miles) to preparation for a half-marathon. How many times longer must you train? How many times more training mileage do you need?

19. *A fly and two trains* (HRW section **2**–4)

Two train locomotives, each traveling at speed 20 miles per hour, are bearing down upon each other on the same long straight track. When the two locomotives are 40 miles apart, a fly traveling at a steady 30 miles per hour leaves from one locomotive. It flies to the other, then back to the first, and continues to zig-zag back and forth between locomotives until the fly and both trains are destroyed in a tragic collision. What is the total distance covered by the fly in its back-and-forth travels? (This is a famous brain teaser. See Norman Macrae, *John von Neumann*, Pantheon Books, New York, 1992, pages 10–11.)

20. *Speed and travel time* (HRW section **2**–4)

It is sometimes claimed that, when driving a specified route, a 10% increase in speed will result in a 10% decrease in travel time. When I first heard this claim, I found it suspiciously general: would an increase of speed from 50 mph to 55 mph really have the same relative effect as an increase from 70 mph to 77 mph? Use calculus to show that this claim is indeed correct to high accuracy.

21. *Road trip*

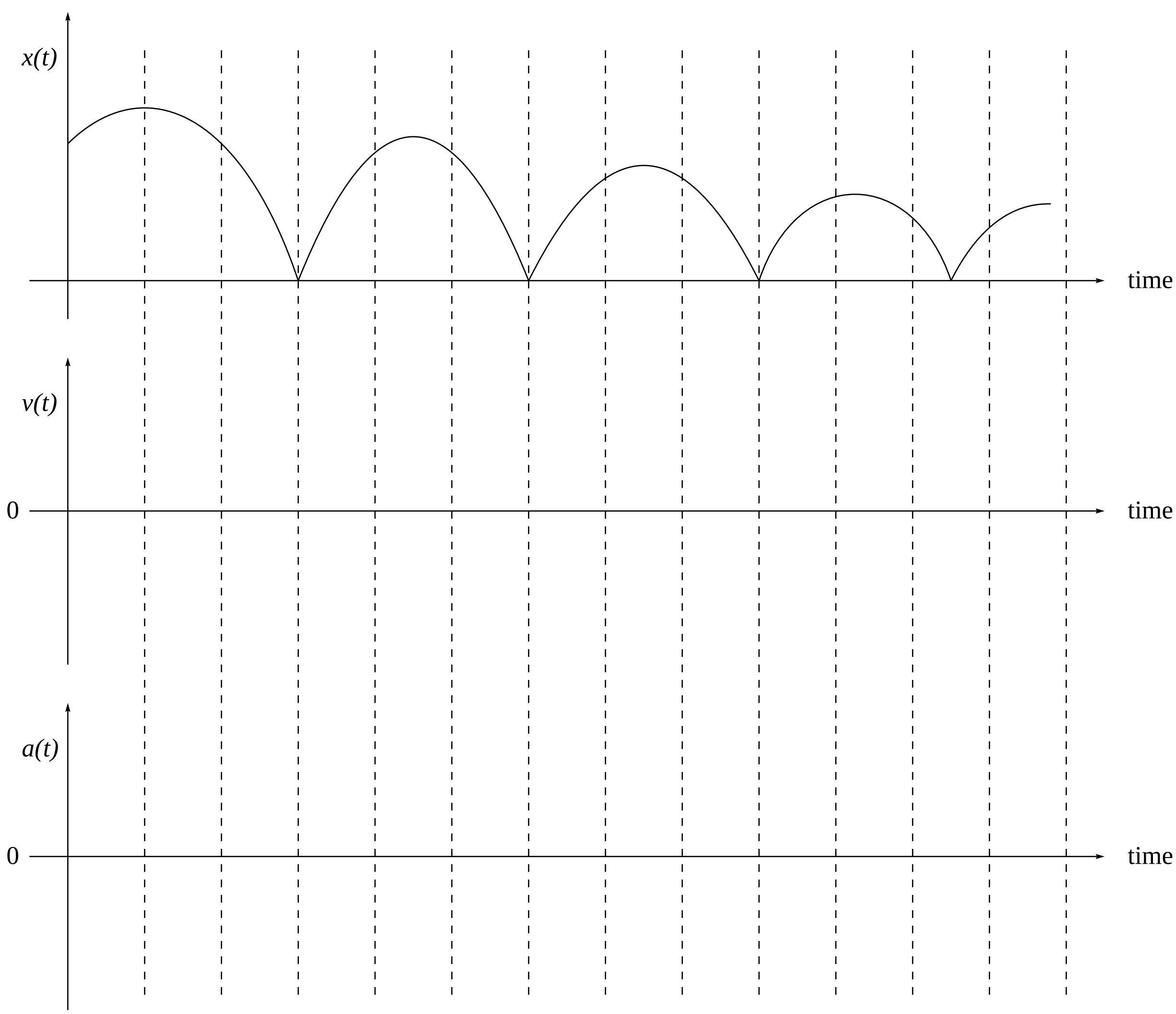
You are driving from Oberlin to Nashville, Tennessee, with three friends. The trip is long and boring. Hank says "If we increase our speed by 10%, it will take about 10% less time to arrive." Beth says "No, it's even better. If we increase our speed by 10%, it will take about 20% less time to arrive." Susan says "You can't be right, Hank, because if that were true then increasing our speed by 100% (that is, doubling our speed) would decrease our travel time by 100% (that is, we'd be there now)."

Which statement is correct? Why? Why are the other two wrong?

22. *Track running* (HRW section **2**–4)

A sign at the Heisman Club Field House announces that 14 laps around the outer edge of the track is the same distance as 15 laps around the inner edge. One day I was jogging around the outer edge of the track when a young woman (one of my students) ran past me on the inner edge. After I had jogged $2\frac{1}{2}$ laps, she passed me again. What was the ratio of her speed to mine?

23. *Bouncing ball* (HRW section **2**–6)
The graph below shows position as a function of time for a bouncing ball. Sketch the velocity and the acceleration as functions of time in the space provided. Your two sketches should not be quantitatively precise, but they must be qualitatively appropriate (with, for example, slopes of the correct sign) and they must accurately locate important features such as zeros, maxima, and minima. Using five or fewer sentences, comment on the significant features of this graph. (Your comments should be both clear and cogent. . . I will grade for quality not quantity.)



24. *Proton in motion* (HRW section **2**–6)
A proton moves along the x axis according to the equation

$$x(t) = (15\ \text{m/s})\, t + (10\ \text{m/s}^2)\, t^2.$$

(The numerical data are accurate to two significant digits.) Calculate (a) the average velocity of the proton during the first 3.0 seconds of its motion, (b) the instantaneous velocity of the proton at $t = 3.0$ s, and (c) the instantaneous acceleration of the proton at $t = 3.0$ s. (d) Sketch a graph of x versus t and indicate how the answer to (a) can be obtained from your graph. (e) Indicate the answer to (b) on the same graph. (f) Sketch a graph of v versus t and indicate on it the answer to (c).

25. *Car vs. rattlesnake*
The head of a striking rattlesnake can accelerate at 50 m/s^2. If a sports car had this performance, how long would it take to reach a speed of 100 km/hr from rest? (All figures reported are significant.)

26. *Bullet*
A bullet leaves a rifle with a 1.20 m-long barrel at speed 640 m/s. If the acceleration within the barrel is constant, how much time does the bullet spend in the barrel?

27. *Cannon shot* (HRW section **2**–7)
In Jules Verne's novel *From the Earth to the Moon*, a projectile fired from a 112.3 meter cannon emerges at the speed of sound, 340.3 m/s. If the acceleration of the projectile was constant, how much time did it spend within the barrel of the cannon? [*Answer:* 0.6600 second — note four significant figures.]

28. *Starting and stopping*
According to "Edmunds Inside Line" (25 March 2009) the 2010 Toyota Prius, while starting up, goes from 0.0 mph to 60.0 mph in 10.1 seconds and, while slowing down (by braking), goes from 60.0 mph to 0.0 mph over 118 feet. Find the acceleration (assumed constant) during start up and slow down. (I found the answer in mph/second, but you may use whichever units seem most appropriate to you.)

Moral of the story: We call the gas petal "the accelerator," despite the fact that the brake petal results in a larger magnitude of acceleration.

29. *Jetliner takeoff* (HRW section **2**–7)
If you think about your experiences being "tossed back into your seat" in a jetliner that's accelerating on a runway, you'll realize that all commercial jetliners accelerate at approximately $g/5$.

a. A Boeing 737 jetliner has a takeoff speed of 70 m/s. How far down the runway must it travel before taking off?

b. The proposed Boeing 7J7 jetliner would have the same runway acceleration as a 737, but a takeoff speed twice that of the 737. Would it need runways twice as long?

30. *Rain speed* (HRW section **2**–9)
On Friday 27 September 2002 the Oberlin area was hit by rains from the remnants of Hurricane Isidore. At 9:43 am, the Lorain County Regional Airport reported calm air (i.e. no wind), but heavy rains from clouds 6 kilometers high. If a raindrop fell with no air resistance from this altitude, at what speed would it hit the ground? In one or two sentences, comment upon the quality of the "neglect air resistance" approximation.

31. *The acceleration of gravity* (HRW section **2**–9)
Express the acceleration of gravity $g = 9.8$ m/s^2 in terms of (miles/hour)/second. If a ball is released from rest, how fast will it be traveling (in miles/hour) in two seconds? How long will it take to exceed a speed limit of 65 miles/hour?

32. *Dropped from Peters Hall* (HRW section **2**–9)
The astronomy observing deck atop Peters Hall is the highest easily-accessible platform at Oberlin College. Suppose a pencil is dropped from rest from the observing deck. How long will it take to hit the ground? (Please don't do this experiment — you might hurt someone.)

a. Find an algebraic answer in terms of the height (call it h) and the acceleration of gravity g.

b. Check your answer: Does it have the dimensions of time? Does it increase with h and decrease with g, as you would expect? Does it give the proper result when $h = 0$? When $g = 0$?

c. Estimate the height of the Peters Hall observing deck. (Suggestion: Measure the height of one sandstone block, then count the number of courses of blocks up to the observing deck.) Numerically evaluate the time required for a pencil to fall from the deck. Does this number seem reasonable to you?

d. How fast will the pencil be going (in miles/hour) when it hits the ground? (Now you understand why I discourage you from performing the experiment!)

33. *Dropping time* (HRW section **2**–9)
Your friend George is an intelligent and thoughtful individual with a vigorous interest in the world around him, but he has one personality quirk: a stringent animosity towards all mathematical equations. One day, you and George discuss falling objects:

> **George:** A ball dropped from a 20-foot-tall building will fall twice as far as a ball dropped from a 10-foot-tall building. Since it goes twice the distance, it will take twice as much time to hit the ground.
>
> **You:** I'm sorry, George, but the equation from part (a) of the previous problem shows that the time required increases like the square root of the height, so the ball will actually take 1.414 times longer to drop from the higher building.
>
> **George:** Don't give me any of that "square root" business. . . you know how I feel about fuzzy math! I say that doubling the drop distance will double the time required!

Prove George wrong with an argument that doesn't use formal mathematical equations, yet which decisively demonstrates that doubling the drop distance will *less than* double the time required. Be firm yet polite: do not endanger your friendship. (One possible argument involves compare these three times: (1) the time to drop 10 feet from a 10-foot building; (2) the time to drop the top 10 feet from a 20-foot building; (3) the time to drop the bottom 10 feet from a 20-foot building.)

34. *New River Gorge jump* (HRW section **2**–9)
Noah Adams's elegant book, *Far Appalachia: Following the New River North* (Delacorte Press, New York, 2001) describes "Bridge Day," when two lanes of West Virginia's New River Gorge bridge are closed to traffic and turn into a unique street festival:

> Parachute jumpers, hundreds of them, will be leaping off these ramps and into the airspace of the New River Gorge. The altitude of the bridge is 1,586 feet [above sea level]. The altitude of the sandbar where the jumpers try to land is 710 feet. If the parachute were not to open, the fall would be eight seconds long . . . and the impact speed 163 miles per hour.

Assuming that these elevations are correct, check the fall duration and impact speed. What unstated assumptions are made?

35. *How high is heaven?* (HRW section **2**–9)
In *Paradise Lost*, Milton describes the fall of Satan from heaven to earth:

> . . . from Morn
> To Noon he fell, from Noon to dewy Eve,
> A Summer's day; and with the setting Sun
> Dropt from the Zenith like a falling Star . . .

It is clear that air resistance can be ignored in this trip, which was mostly through outer space. If we assume that Satan started from rest and fell with constant acceleration g for a full "Summer's day" (17.5 hours of daylight), how high is heaven?

36. *Falling brick* (HRW section **2**–9)
A brick dropped from rest falls 9.0 meters before landing on a pillow. The pillow compresses 14.1 cm and then the brick stops. What is the acceleration (assumed constant) of the brick while it is in contact with the pillow?

37. *Dropping time in an imaginary universe* (HRW section **2**–9)
Here is a physics problem that you are *not* supposed to solve:

> Suppose that in J.R.R. Tolkien's imaginary universe of Middle-earth, objects fall not with constant acceleration $d^2y/dt^2 = -g$ but with constant "jerkiness" $d^3y/dt^3 = -j$ (positive direction taken upward). During a battle in the mines of Moria (book II, chapter 5), an orc falls from rest down a well of depth d. How long does it take for the orc to reach the bottom of the well?

Four friends work this problem independently. When they get together afterwards to compare results, they find that they have produced four different answers! The candidate answers are

(a) $7d + j$

(b) $d - 7j$

(c) $\sqrt[3]{6d/j}$

(d) $(2d/j)^2$

Provide simple reasons showing that three of these candidate answers must be incorrect. [*Answers:* Candidates (a), (b), and (d) all have incorrect dimensions. Candidate (b) can give negative values for the drop time. The drop time ought to decrease as j increases — candidate (a) fails this test. Candidates (a) and (b) fail for the special cases $d = 0$ (which should give $T = 0$) and $j = 0$ (which should give $T = \infty$).]

38. *An erroneous claim* (HRW section **2**–9)
A ball tossed vertically into the air at speed v_0 reaches a height of $H = v_0^2/(2g)$. I once saw this result quoted in a book "upside down": that is, the book claimed that the height reached was $H = 2g/v_0^2$. Give at least five reasons why the book's claim must be erroneous. (Deriving the correct result counts as one reason, but there are many easier and more direct ways to prove the book's result wrong.)

[*Answer:* The claim is that H is given by $2g/v_0^2$, but in fact:

1. This claim has the dimensions of [1/(meters)] not of [meters].
2. This claim would have H increase when v_0 decreases.
   (That is, higher tosses come from slower throws!)
3. This claim would have H increase when g increases.
   (That is, higher tosses on the Earth than on the Moon!)
4. This claim would have $H = \infty$ when $v_0 = 0$.
   (That is, a ball dropped from rest would go through the roof!)
5. This claim would have $H = 0$ when $g = 0$.
   (That is, you could never toss a ball up on the space shuttle!)]

39. *Dropping with air resistance* (HRW section **2**–9)
Here is a physics problem that you are *not* supposed to solve:

> A pencil is dropped from rest from a building of height h. Because of air resistance, the acceleration is
> $$a = -g - kv,$$
> where v is the velocity and k is a constant. How long will it take to hit the ground?

Four friends work this problem independently. When they get together afterwards to compare results, they find that they have produced four different answers! The candidate answers are

$$t_D = \sqrt{\frac{2h}{g} + \frac{k}{3}\left(\frac{2h}{g}\right)^{3/2}} \tag{3.1}$$

$$t_D = \sqrt{\frac{h}{g} + \frac{k}{3}\left(\frac{2h}{g}\right)^{3/2}} \tag{3.2}$$

$$t_D = \sqrt{\frac{2h}{g} + \frac{k}{3}\left(\frac{h^2}{g}\right)^{3/2}} \tag{3.3}$$

$$t_D = \sqrt{\frac{2h}{g} - \frac{k}{3}\left(\frac{2h}{g}\right)^{3/2}} \tag{3.4}$$

Without solving the problem, provide simple reasons showing that three of these candidate answers must be incorrect. (The remaining candidate is also incorrect, but it's a good approximation.)

40. *Flying apple* (HRW section **4**–6)
The Professor Street bridge over Plum Creek has a guard rail at a height of 3.2 m above the water surface. My son once placed an apple on the guard rail, then flipped it off using a hockey stick so that the apple flew off with speed 3.7 m/s at an angle of 32° above the horizontal. What was the apple's speed when it splashed into the water?

[*Answer:* 8.7 m/s. I gave this problem in the first exam in 2003. Many students found it difficult because they split the problem into stages: For example they first solved the problem from launch to the peak of the trajectory, and then used the solution for that stage to solve the problem from the peak to the splash. Or they solved it from launch to peak, then from peak to the level of the guard rail, and then from the level of the guard rail to the splash. Breaking up the problem into stages like this is completely unnecessary — the apple is in uniform free fall from the moment it leaves the hockey stick — and it makes the solution both more difficult and more subject to error.]

41. *Cliff toss*
A baseball tossed from the top of a cliff lands a distance R from the bottom of the cliff after time T has elapsed. Find a formula for the initial speed in terms of R, T, the cliff height h, and the acceleration of gravity g. Then find a formula for the angle of the toss (relative to the horizontal).

42. *Angle of flight* (HRW section **4**–6)
A ball is launched at speed v_0 with angle (relative to horizontal) θ_0. Show that when it attains an elevation of y above the launch point the angle of flight is θ where

$$\tan^2\theta = \tan^2\theta_0 + \frac{2gh}{v_0\cos^2\theta_0}.$$

43. *Long jump* (HRW section **4**–6)
Given that the world record time for the 100 meter sprint is about 10 seconds, estimate the world record distance for the long jump. What factors are you neglecting in this estimate? (The actual world record long jump was 8.95 m, performed in 1991 by the American Mike Powell at Tokyo.)

44. *Artemis in a pickup truck* (HRW section **4**–6)
Here is a physics problem that you are *not* supposed to solve:

> A pickup truck travels east on a level plane at speed v_T. In the back of the truck an archer strings her bow, aims forward (eastward) at angle θ above the horizon, and lets fly an arrow with speed v_0 relative to the truck. How far from the launch point will the arrow strike the ground?

Four friends work this problem independently. When they get together afterwards to compare results, they find that they have produced four different answers! The candidate answers are

(a) $(2v_T v_0^2 \sin\theta + v_0^2 \sin 2\theta)/g$

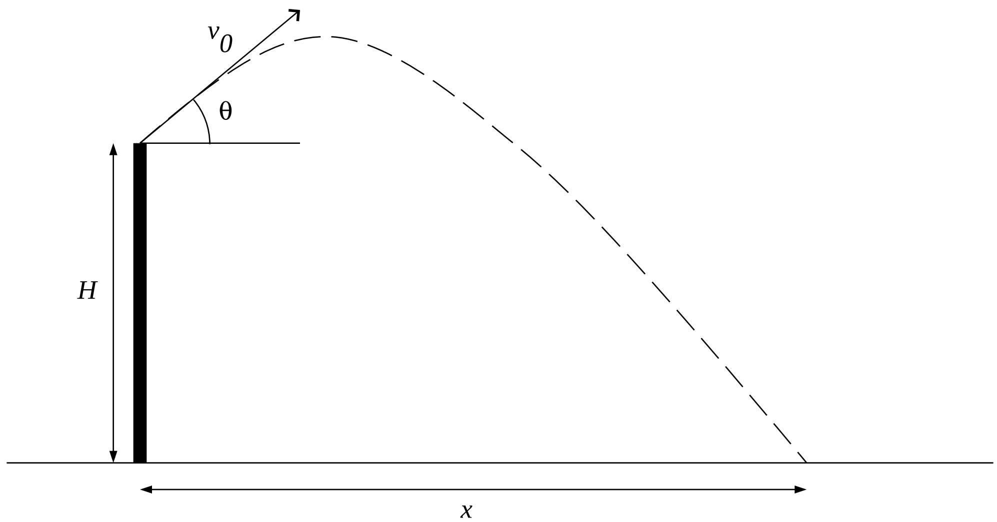
(b) $(2v_T v_0 \sin\theta + 2v_0^2 \sin 2\theta)/g$

(c) $(2v_T v_0 \cos\theta + v_0^2 \sin 2\theta)/g$

(d) $(2v_T v_0 \sin\theta + v_0^2 \sin 2\theta)/g$

Without solving the problem, provide simple reasons showing that three of these candidate answers must be incorrect.

45. *Fire tower* (HRW section **4**–6)
Here is a physics problem that you are *not* supposed to solve:



> An archer stands atop a fire tower of height H and shoots an arrow with speed v_0 at an angle θ above the horizontal. How far from the base of the tower does the arrow land?

Four friends work this problem independently. When they get together to compare answers, they find that they have produced four different results! These are:

$$x = \frac{1}{g}\left[v_0^2 \sin\theta\cos\theta + v_0\cos\theta\sqrt{v_0^2\sin^2\theta + 2gH}\right] \tag{3.1}$$

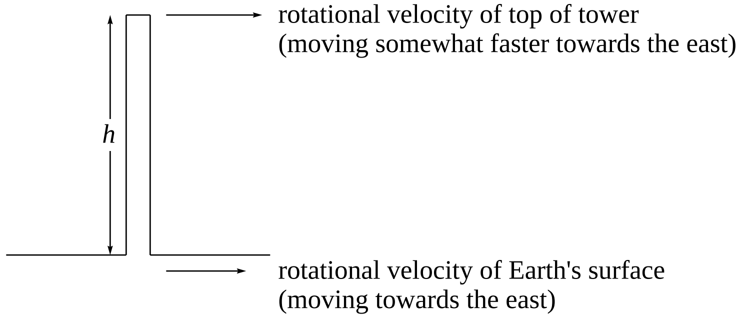
$$x = \frac{1}{g}\left[v_0^2 \sin\theta\cos\theta + v_0\cos\theta\sqrt{v_0^2\sin^2\theta - 2gH}\right] \tag{3.2}$$

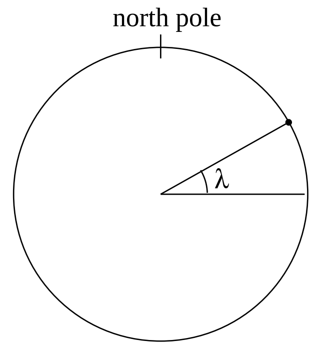
$$x = \frac{1}{g}\left[v_0^2 \sin\theta\tan(\theta/2) + v_0\cos\theta\sqrt{v_0^2\sin^2\theta + 2gH}\right] \tag{3.3}$$

$$x = \frac{1}{g}\left[v_0^2 \sin\theta\cos\theta + v_0\cos\theta\sqrt{v_0\sin^2\theta + 2gH}\right] \tag{3.4}$$

Show that three of these four are unacceptable on physical grounds, then prove that the remaining candidate is, indeed, correct. [*Answer:* The arrow should travel farther from a higher tower, but equation (2) has x decreasing as H increases (also, for sufficiently high towers x will be complex-valued!). If $\theta = \pm 90°$, the arrow should land at $x = 0$, which is not the result given by equation (3). Finally, equation (4) demonstrates the ever-popular inconsistent dimensions.]

46. *Tower drop* (HRW section **4**–6)
Because of the rotation of the Earth, a ball dropped from a tower will not land directly below its launch point, but rather slightly to the east. (This effect is called "the Coriolis pseudoforce".)





How far to the east? Here are five candidate solutions. Provide simple reasons showing that four of these candidates must be incorrect. (The height of the tower is h, and it is located at latitude λ. The rotational period of the Earth is $T = 24$ hours $= 86\,400$ sec.)

$$\frac{2\pi}{3}\frac{\cos\lambda}{T}\sqrt{\frac{8h^3}{g^2}} \tag{3.1}$$

$$\frac{2\pi}{3}\frac{\sin\lambda}{T}\sqrt{\frac{8h^3}{g}} \tag{3.2}$$

$$\frac{2\pi}{3}\frac{\cos\lambda}{T}\sqrt{\frac{8h^3}{g}} \tag{3.3}$$

$$\frac{2\pi}{3}(\cos\lambda)T\sqrt{8hg} \tag{3.4}$$

$$\frac{2\pi}{3}(\cos\lambda)T^3\sqrt{\frac{8g^3}{h}} \tag{3.5}$$

47. *Acceleration in uniform circular motion* (HRW section **4**–7)
Your textbook provides a graphical argument showing that, in uniform circular motion, the acceleration is directed towards the center of the circle and has magnitude v^2/r. This problem derives the same result in a more formal, algebraic manner.

For an object in uniform circular motion about the origin, the position as a function of time is given by

$$x(t) = r\cos(\omega t); \qquad y(t) = r\sin(\omega t).$$

(Note: The symbol ω is the Greek letter "omega", not the roman letter "w". It's a physics faux pas to say "doubleyou" when you mean "omega".)

a. The quantity ω is called the "angular velocity" because it's the rate that the polar angle θ changes with time. Show that it is related to the period T through $\omega = 2\pi/T$. Thus the angular velocity has the dimensions of "[radians]/[time]" or (because radians are dimensionless) "1/[time]". (The phrase "$\omega = 2.73$ sec^{-1}" is pronounced "omega is 2.73 inverse seconds".)

b. Differentiate the expressions for $x(t)$ and $y(t)$ to find $v_x(t)$ and $v_y(t)$. Show that the magnitude of the velocity is $v = \omega r$.

c. Check the preceding result: Does it have the right dimensions? Does it make sense that the speed would increase with increasing r? That it would decrease with increasing T?

d. Differentiate again to find $a_x(t)$ and $a_y(t)$. Show that the acceleration vector is

$$\vec{a} = -\omega^2 \vec{r} = -\frac{v^2}{r^2}\vec{r}.$$

(Here $\vec{r}$ represents the position vector, whereas r represents the length of the position vector.)

48. *Cars on curves* (HRW section **4**–7)

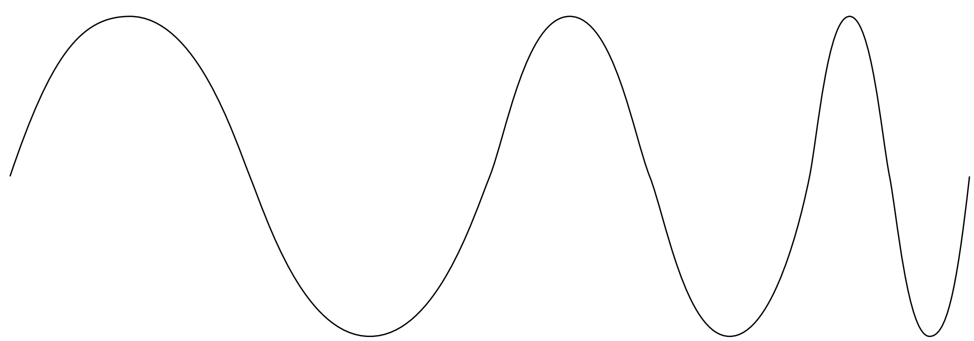
A blue car takes a hairpin curve at 20 miles per hour. A red car takes the same curve at 40 miles per hour. Does the red car, moving twice as fast, experience twice the acceleration of the blue car? If not, then what is the ratio of accelerations? Identify any assumptions you've made in your analysis.

49. *Wide circles* (HRW section **4**–7)

The acceleration of an object undergoing uniform circular motion is v^2/r. Discuss this equation in the limit $r \to \infty$.

50. *Winding road* (HRW section **4**–7)

A car travels on the winding road below at constant speed. Sketch the acceleration vectors.



51. *Exploding star* (HRW section **4**–7)

In one stage in stellar evolution, a rotating star shrinks and the speed of its rotation increases. This process continues until material at the star's surface rips away. During the shrinking process the product of the star's radius and its equatorial surface velocity remains constant. (We will see later that this is due to the conservation of angular momentum.) The surface material rips away when it experiences a threshold acceleration a_{rip}.

Find a formula for the radius at which material first rips off in terms of a_{rip}, the initial radius r_0, and the initial equatorial surface velocity v_0.

52. *Jerkiness in uniform circular motion* (HRW section **4**–7)
You know that in uniform circular motion, the acceleration $\vec{a}(t)$ is not constant, because if the acceleration were constant then the trajectory would be a parabola. But what exactly is the jerkiness

$$\vec{j}(t) = \frac{d\vec{a}(t)}{dt}?$$

Given the statement "If $\vec{r}(t)$ moves uniformly in a circle at speed

$$v = |\vec{v}(t)| = \left|\frac{d\vec{r}(t)}{dt}\right|,$$

then the acceleration is

$$\vec{a}(t) = -\frac{v^2}{r}\hat{r}(t)\text{ ”},$$

show immediately that

$$\vec{j}(t) = -\frac{v^3}{r^2}\hat{v}(t).$$

53. *Science education standards* (HRW chapter **5**)

a. According to the State of Ohio's former science education standards (*Fourth-grade Proficiency Tests: Information Guide*, Ohio Department of Education, Columbus, Ohio, August 1995, page 57)

> Among the fundamental concepts [fourth-grade] students should understand are that [1] things move only when something moves them; [2] they keep moving until something stops them; [3] the harder something is pushed, the faster it goes; and [4] the more massive something is, the harder it is to move.

Laying aside any concerns about the *appropriateness* of this standard (Is this the sort of item that *every person* ought to know? Is this topic developmentally appropriate for fourth-graders?), comment simply on the *correctness* of this passage.

b. When the State of Ohio revised its science education standards, the Fall 2001 draft standards claimed that Kindergarten students should

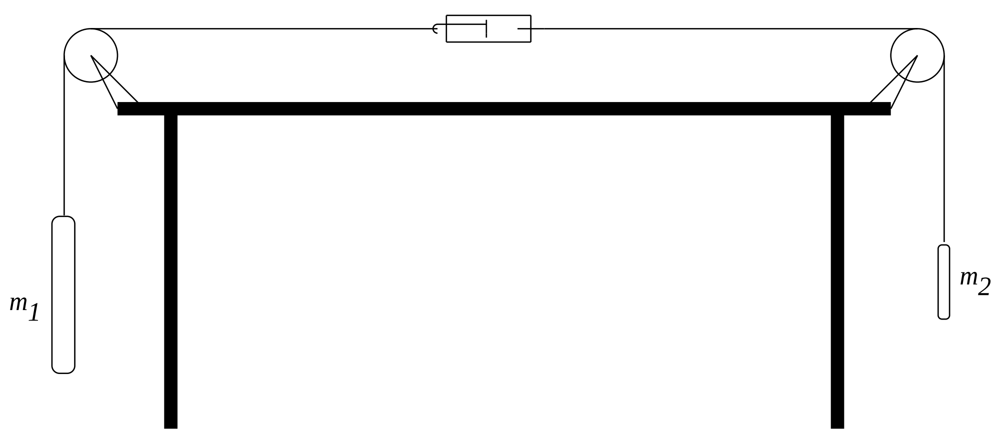
> Know that objects change how they are moving when pushed or pulled.

Yet a brick, at rest on a table, can be pushed downward without changing its motion. (It is at rest both before and after experiencing the downward push, so its motion doesn't change.) What essential piece of $\sum \vec{F} = m\vec{a}$ is omitted from this sentence? (Fortunately, this error in science was caught while the standards were still in draft form. Once again, leave aside the question of whether Kindergarten is the appropriate place to learn this concept.)

54. *Snow cushion* (HRW section **5**–6)

In February 1955, a paratrooper on a training jump fell 370 m from an airplane. Unfortunately, due to mechanical failure his parachute didn't open. Fortunately, he landed in snow and suffered only minor injuries. Assume that his speed at impact was 56 m/s (terminal speed), that his mass (including gear) was 85 kg, and that the magnitude of the force on him from the snow was a constant 1.2×10^5 N (the survival limit). How deeply did he plow into the snow before stopping?

55. *Sliding salami* (HRW section **5**–9)



A salami of mass m_1 is attached to one end of the cord shown above, and a salami of mass m_2 is attached to the other end. The masses of the cord and of the spring scale are negligible compared to the masses of the salamis. The more massive salami falls and the less massive salami rises until the less massive salami reaches and then jams its pulley.

a. Find the tension in the cord while the salamis are falling and rising. (That is, before the less massive salami jams its pulley.) (This is the tension that will register in the spring scale if it has stopped oscillating.)

b. If m_1 and m_2 are interchanged, the string will still carry the same tension. Does your equation reflect this symmetry?

c. Check your result in the special cases $m_1 = m_2$, $m_1 = 0$, and $m_2 = 0$.

Answer:

$$T = 2g\left(\frac{m_1 m_2}{m_1 + m_2}\right).$$

56. *A girl, a sled, and an ice-covered lake* (HRW section **5**–9)

An 8.4 kg sled lies on the frictionless ice of a frozen lake, connected to a 40 kg girl on shore by a 15 m rope. The mass of the rope is much less than the mass of the girl or of the sled, so we call its mass "negligible." The girl pulls on the rope with a horizontal force of 5.2 N. (Throughout this problem, only horizontal forces and motions are relevant, so you may ignore the vertical dimension entirely.)

a. What forces act on the sled? What is the acceleration of the sled?

b. What forces act on the girl? What is the acceleration of the girl?

After drawing the sled to shore, the girl kicks it back out onto the ice again and then steps out onto the ice herself. Again she pulls on the 15 m rope with a horizontal force of 5.2 N.

c. What forces act on the sled? What is the acceleration of the sled?

d. What forces act on the girl? What is the acceleration of the girl?

e. How far from the girl's initial position do the girl and sled meet?

f. In part (b), you found equal and opposite forces acting on the girl. In parts (c) and (d), you found equal and opposite forces acting on the sled and on the girl. In which case is the pair of forces equal and opposite because of Newton's third law, and in which case is the pair equal and opposite because of Newton's first law?

57. *Science text* (HRW chapter **5**)

When my younger son was in sixth grade, his science textbook described friction as "a force, usually produced by rubbing, that resists motion." (*Holt Science*, page 38). Comment.

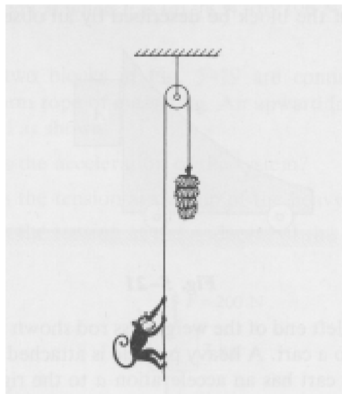
58. *Monkey business* (HRW section **5**–9)

A lightweight cord passes over a frictionless pulley. Hanging on one end of the cord is 33 kg of bananas. On the other end of the cord is a 22 kg monkey. The monkey pulls himself up with cord with such a large acceleration that the bananas are stationary.

a. What is the monkey's acceleration?

b. If the masses of the monkey and bananas both doubled, what would the new acceleration be?

59. *The hungry monkey* (HRW section **5**–9)
(From F.W. Sears, M.W. Zemansky, and H.D. Young, *University Physics*, fifth edition, 1976, page 88.) A fifty-pound monkey with downcast eyes has a firm grip on a light rope that passes over a frictionless pulley and is attached to a fifty-pound bunch of bananas.



The monkey lifts his eyes, spies the bananas, and starts to climb the rope to get to them.

a. As he climbs, do the bananas move up, down, or remain at rest?

b. As he climbs, does the distance between the monkey and the bananas decrease, increase, or remain constant?

c. The monkey releases his grip on the rope. What happens to the distance between the monkey and the bananas now?

d. Before hitting the ground, the monkey grabs the rope and and stays his fall. What do the bananas do?

60. *Science education materials* (HRW chapter **5**)
When my elder son was in sixth grade, he brought home from school a science worksheet in which two space pilots engage in the following conversation:

> "That was a very strong force we felt during blastoff."
> "Well, this spacecraft has a great deal of inertia that has to be overcome."
> "It sure does."

Comment. In particular, can a force — measured in newtons — ever be greater than an inertia — measured in kilograms? (Another sad aspect of this worksheet was that it discussed velocity, acceleration, inertia, and force, but it was titled "Energy in Space.")

61. *Science test, I* (HRW chapter **5**)
When my elder son was in sixth grade, he was given a "science test" that included the following multiple choice questions:

1. When you sit down on a chair, the force pushing down on the seat is (a) the action force; (b) the reaction force; (c) due to your lift; (d) due to your temperature; (e) gravity.
2. When you sit on a chair, the force of the chair seat pushing up on you is (a) the action force; (b) the reaction force; (c) greater than the force pushing down; (d) less than the force pushing down.
3. Newton's third law of motion explains (a) a person thinking about moving from place to place; (b) an apple falling from a tree; (c) a person jumping on a trampoline and then being pushed upward; (d) the way in which your optic nerve works.

Comment.

62. *Science test, II* (HRW chapter **5**)
When my younger son was in sixth grade, he was given a "science test" that matched the term "thrust" to the description "the forward motion of an airplane". Comment.

63. *Pendulum motion* (HRW chapter **15**)
A pendulum bob of mass m hangs from a string of length L. The pendulum's period T can depend only upon m, L, the acceleration of gravity g, and the amplitude of oscillation θ_{max}. Show that the only way to combine these four variables to produce a function with the dimensions of time is through

$$T = f(\theta_{\text{max}})\sqrt{\frac{L}{g}},$$

where $f(\theta_{\text{max}})$ is an unknown, but dimensionless, function. Can you explain why the period is independent of mass m using an argument more enlightening than "it comes out of the math"?

64. *Oscillator with friction*
A block of mass m moves in one dimension subject to a spring force $-kx$ and a friction force $-bv$ (where x is position and v is velocity). If the friction force is relatively small, then the block's motion is nearly simple harmonic. Which one of the following expressions is a plausible candidate for the period of that motion? Justify your choice.

$$2\pi\sqrt{\frac{m}{k}} - \frac{\pi}{4}\frac{b^2}{\sqrt{mk^3}} \tag{3.1}$$

$$4\pi\sqrt{\frac{m}{k}} + \frac{\pi}{4}\frac{b^2}{\sqrt{mk^3}} \tag{3.2}$$

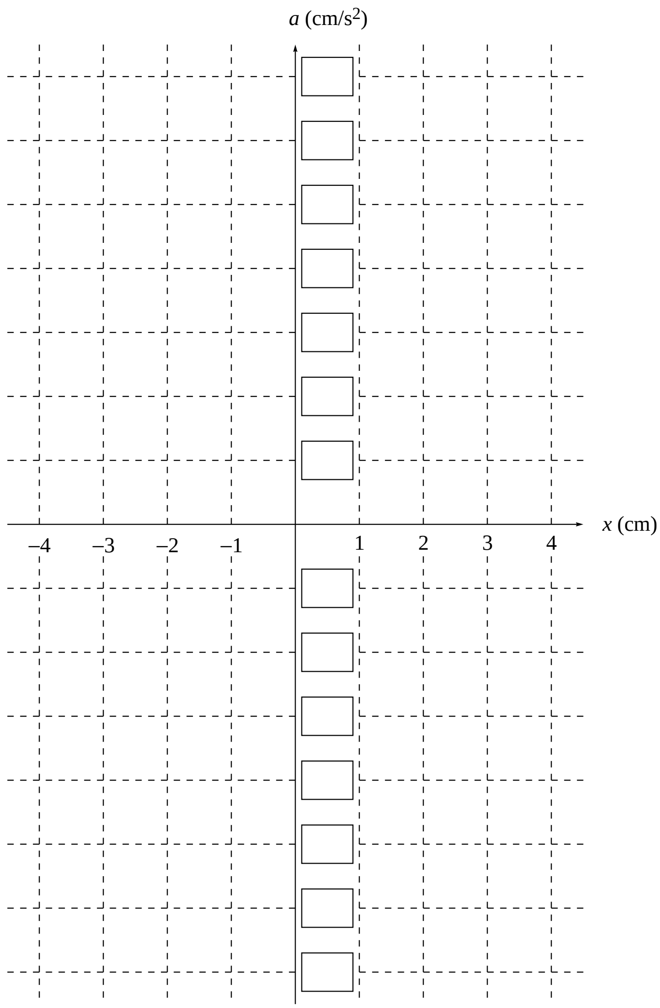
$$2\pi\sqrt{\frac{m}{k}} + \frac{\pi}{4}\frac{b^2}{\sqrt{mk^2}} \tag{3.3}$$

$$2\pi\sqrt{\frac{m}{k}} + \frac{\pi}{4}\frac{b^2}{\sqrt{mk^3}} \tag{3.4}$$

$$2\pi\sqrt{\frac{m}{k}} + \frac{\pi}{4}\frac{b^2 + mk}{\sqrt{mk^3}} \tag{3.5}$$

65. *Simple harmonic motion graph, I* (HRW section **15**–2)
Sketch v as a function of x for simple harmonic motion. In three or fewer sentences, explain how you can use your sketch to find the amplitude and period of the motion.

66. *Simple harmonic motion graph, II* (HRW section **15**–2)
A bob executes simple harmonic motion with amplitude 3.00 cm and period 3.14 s. (a) What is the bob's maximum acceleration? (b) Plot acceleration versus position on the grid below. Be sure to fill in numbers within the boxes so as to calibrate the vertical axis.

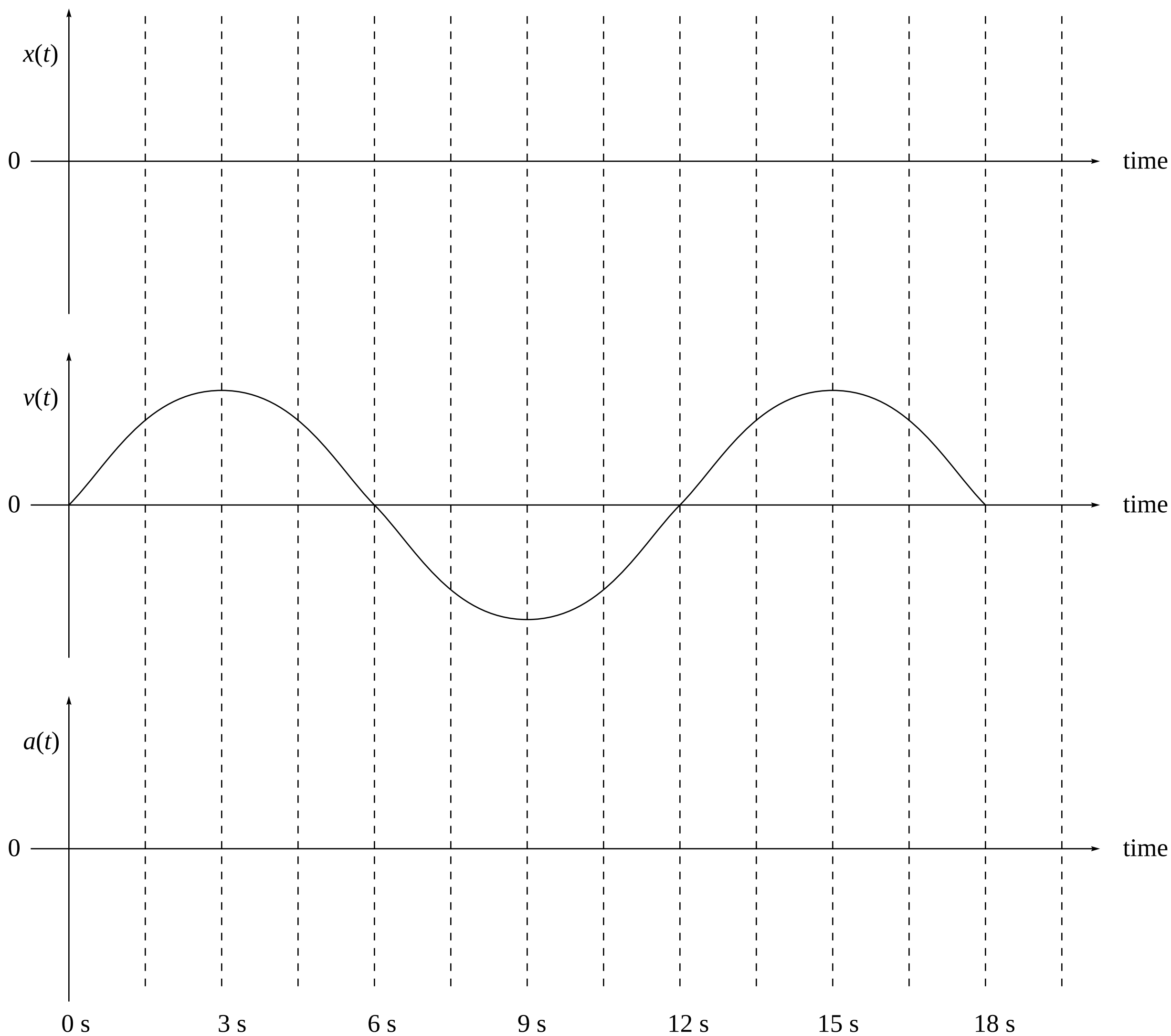


67. *Simple harmonic motion graph, III* (HRW section **15**–2)
The graph below shows velocity as a function of time for a simple harmonic oscillator.

a. What is the period of the oscillator? [2 points]

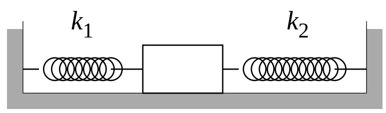
b. Sketch the position and the acceleration as functions of time in the space provided. Pay particular attention to points where the curves cross zero or take on maximum or minimum values. (Be sure to double check the signs!) [4 points]

c. If the maximum velocity is 3.5 m/s, what is the maximum acceleration? [4 points]



68. *A block and two springs*

Two springs are attached to a block and to fixed supports as shown in this figure. The block rests on a frictionless surface.



The spring on the left has spring constant k_1, and if the block is attached to the left spring alone it oscillates with frequency f_1. The spring on the right has spring constant k_2, and if the block is attached to the right spring alone it oscillates with frequency f_2. Show that if the block is attached to both springs, it oscillates with frequency

$$f = \sqrt{f_1^2 + f_2^2}.$$

69. *Quark-quark interactions* (HRW section **7**–8)

According to the theory of quantum chromodynamics (QCD), protons and neutrons are made up of quarks, and the force between two quarks separated by a distance x is approximately

$$F_Q(x) = -F_\infty \left(1 + \frac{\Lambda}{x}\right),$$

where F_∞ and Λ are positive constants.

a. Sketch this force function and compare it qualitatively to the gravitational force function $F_G(x) = -k/x^2$.

b. One quark is fixed at the origin and a second moves from x_i to x_f. How much work does the QCD force do during this motion?

⟦For background information, see L.P. Fulcher, "Perturbative QCD, a universal QCD scale, long-range spin-orbit potential, and the properties of heavy quarkonia," *Physical Review* D, **44** (1991) 2079–2084. This interaction is called the "Richardson potential".⟧

70. *Work with force preconceptions* (HRW chapter **7**)

While in truth the net force on a particle results in the acceleration of that particle, many non-physicists hold the naïve preconception that net force results in velocity:

$$\begin{aligned} \text{correct:} &\quad \sum \vec{F} = m\vec{a} \\ \text{naïve preconception:} &\quad \sum \vec{F} = m\vec{v}. \end{aligned}$$

Show that if the naïve preconception were correct, then the net work done on *any* particle under *any* circumstances would always be positive or zero.

71. *Skier* (HRW section **8**–5)

A skier of mass 83.4 kg starts from rest and takes the "Ruby Roller Coaster" route: this route descends 12.3 m, then ascends 6.5 m, then turns left and descends 3.5 m, then ascents 4.4 m, then turns right and descends 10.7 m to a level run.

a. If the skier doesn't push himself with his poles, and if friction is negligible, what is his final speed?

b. This route is also executed by a skier of mass 64.5 kg. What is her final speed?

72. *Sledding with a push* (HRW section **8**–5)

Your friend George (from additional problem 33) finds that if he starts from rest and sleds down a certain snow-covered slope, he reaches the bottom going 14 miles per hour. He reasons that if a friend were to push him so that he starts at 5 miles per hour, then he would attain the speed of 19 miles per hour ($14 + 5 = 19$).

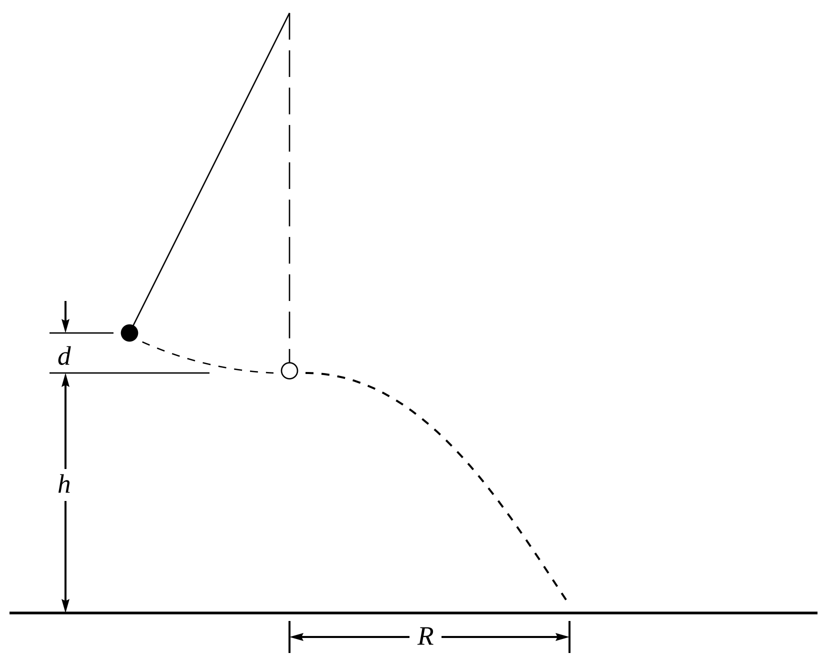
a. Use energy conservation to show that, in fact, when George starts at 5 miles per hour his final speed increases to 15 miles per hour.

b. George doesn't understand square roots, and he thinks that energy is an ethereal fluid produced by the Great Pyramid and channeled by quartz crystals. Produce an argument that George would understand demonstrating that the final speed is *less* than 19 miles per hour. *Clue:* Does he spend the same amount of time on the slope after a push start?

73. *Triplets on a trampoline* (HRW section **8**–5)

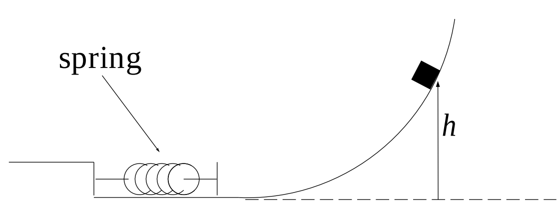
Three children — identical triplets each of mass 17 kg — play on a trampoline. One child sitting on the middle of the trampoline depresses it by 0.021 meter, while two children depress it by 0.042 meter and three by 0.063 meter. All three children get off the trampoline, and then one child climbs up on a tree limb 2.2 meters above the trampoline and drops off. How far does the trampoline depress before the child stops? (Assume that this depression is much less than 2.2 meters.)

74. *Pendulum challenge* (HRW section **8**–5)
In the "Pendulum Challenge" workshop laboratory, a lead ball swings down an arc as a pendulum bob, but at the bottom of the arc it is released from its cord and begins freefall with a finite horizontal velocity.



Find an expression for the range R in terms of the heights d and h. ⟦Bonus: Remarkably, the range is independent of the acceleration of gravity g: that is, the ball would have the same range whether the experiment were performed on the Earth, the Moon, or Jupiter. (Of course, the process would take more time on the Moon and less time on Jupiter, but the ball would follow exactly the same trajectory on each planet.) Can you produce any argument (other than "it come out of the mathematics") to show that this should be true?⟧

75. *Slide* (HRW section **8**–5)



A block released from rest slides down a frictionless slope until it comes to rest against a horizontal spring backstop. If the same block is released from twice the initial height, will it compress the spring twice as much? Explain. Correct the statement if it's wrong.

76. *Spring gun* (HRW section **8**–5)
(This problem was inspired by HRW problem 8–34.) Here is a physics problem that your are *not* supposed to solve:

> A horizontal spring gun of spring constant k is mounted a distance H above the floor. If the spring is compressed by distance d, how far from the launch point does it land?

Here are five different candidate solutions for the problem:

$$x_{\rm hit} = \sqrt{\frac{2kH}{mg}}\, d \tag{3.1}$$

$$x_{\rm hit} = \sqrt{\frac{mg}{2kH}}\, d \tag{3.2}$$

$$x_{\rm hit} = \sqrt{\frac{k(H-d)}{mg}}\, d \tag{3.3}$$

$$x_{\rm hit} = \sqrt{\frac{2kHd}{mg}} \tag{3.4}$$

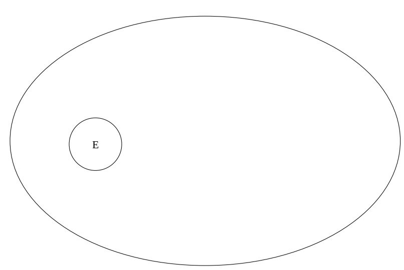
$$x_{\rm hit} = \sqrt{\frac{2kH}{mg^2}}\, d \tag{3.5}$$

Find something wrong with all but one. *Answers:* Candidate (2) has $x_{\rm hit}$ decrease when H increases. . . never! Candidate (3) has $x_{\rm hit} = 0$ when $H = d$. . . no way! Candidates (4) and (5) are dimensionally incorrect.

77. *Jumping* (HRW section **8**–5)
Borelli's law states that animals of similar construction jump to about the same height from a standing start, regardless of the animal's size. (See D'Arcy Wentworth Thompson, *On Growth and Form* (Cambridge University Press, 1942) pages 36–37.) For example, a typical person is about 2000 times taller than a typical flea, yet both jump about two feet high. (This statement is really a "regularity," not a "law," because it isn't always followed: for example a tortoise *cannot* jump about two feet high.)

Assuming that the amount of work done by muscles in a jump is proportional to the amount of muscle, and hence to the mass of the animal, prove Borelli's law.

78. *Satellite speeds* (HRW section **8**–5)



A satellite of mass m follows an elliptical orbit about the Earth. At closest approach to Earth, is it going faster or slower than it was going when farthest away from Earth? Find an equation for the speed at closest approach (v_c) in terms of the speed at farthest away (v_f), the distance at closest approach (r_c), and the distance at farthest away (r_f). Be sure to define any constants you use. (Distances r are measured from the center of the Earth. If you're interested, the point of closest approach is called "perigee," the point farthest away is called "apogee.")

79. *Ice mound, part II* (HRW section **8**–5)

HRW problem 8-36 shows that a boy slipping from the summit of a hemispherical mound of ice (with radius R) will leave the mound's surface when he reaches the height of $\frac{2}{3}R$.

a. One remarkable aspect of this result is that the "launch height" depends only upon R and not upon m or g. That is, the boy will launch free of the mound's surface at the same height whether or not he's wearing a heavy backpack, and the launch height will be the same whether the experiment is performed on the Earth or on the Moon. Can you find a reason for this other than simply "it comes out of the mathematics"?

b. On the other hand, the amount of time between start and launch *does* depend upon both R and g. (It makes sense that the time should be *greater* on the Moon than on Earth.) Without solving the problem completely, show that the formula for the time required has the functional form $C\sqrt{R/g}$, where C is a dimensionless constant.

80. *Ice mound, part III* (HRW section **8**–5)
Here is a physics problem that you are *not* supposed to solve:

> HRW problem 8-36 (page 192) discusses a boy sliding down a hemispherical ice mound and shows that, if the boy's initial speed is zero, then he leaves the mound at height $\frac{2}{3}R$. If the boy's initial speed were instead v_0 at what height would he leave the mound?

Five students work this problem independently. When they get together afterwards to compare answers, they find they have produced five different answers! The candidate answers are:

$$\frac{2}{5}R + \frac{v_0^2}{3g} \tag{3.1}$$

$$\frac{2}{3}R + \frac{v_0^2}{3g} \tag{3.2}$$

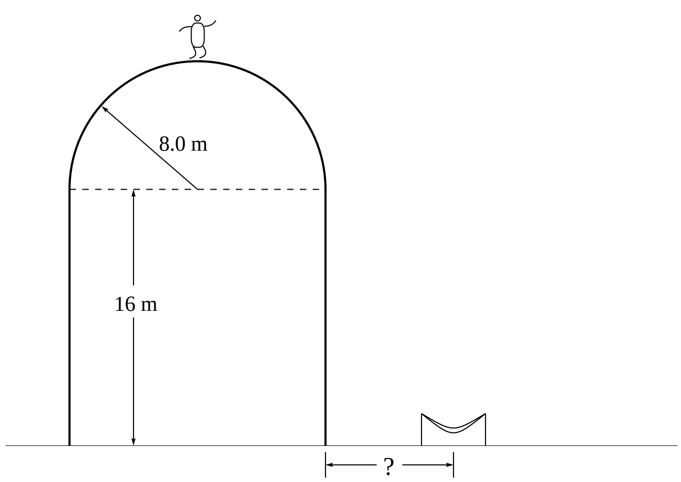
$$\frac{2}{3}R + \frac{v_0}{3g} \tag{3.3}$$

$$\frac{2}{3}R - \frac{v_0^2}{3g} \tag{3.4}$$

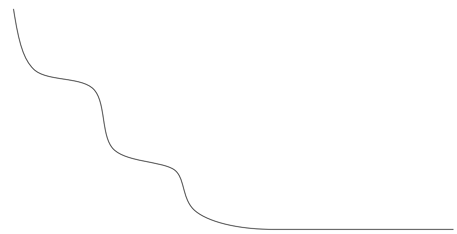
$$\sqrt{\frac{2v_0^2 R}{3g}} \tag{3.5}$$

Find something wrong with all the candidates but one.

81. *Daredevil astronomer* (HRW section **8**–5)
(Modified from a problem in A.P. French, *Newtonian Mechanics*, page 481.) A daredevil astronomer stands at the top of his observatory dome wearing roller skates, and starts with negligible velocity to coast down over the dome's surface. Where should the astronomer's assistant position a net?



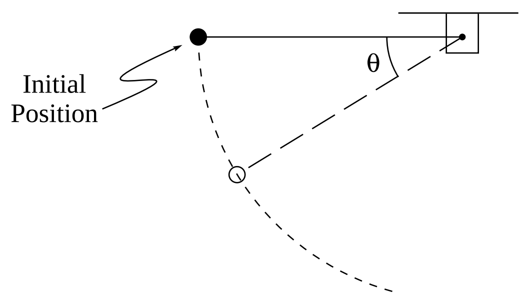
82. *Water slide* (HRW section **8**–5)
The Silverwood Theme Park in Athol, Idaho has a 180-foot-tall water slide shaped like this:



(The steep sections are probably not quite so steep as depicted in the figure, but that's how steep they seemed to me when I was sliding down them.) Silverwood advertising claims that riders will be weightless (that is, the normal force goes to zero) twice during their trip down this water slide. When I heard this advertisement, I said to myself "Nonsense! The normal force depends on the mass of the individual, so different individuals will experience different amounts of weightlessness." But when I tried it out, I found that the advertising was right and my objections were wrong. Explain.

83. *Snow sledding* (HRW section **8**–7)
(I thought of this problem on Wednesday 28 January 2004, when the Oberlin public schools had a snow day so I went out sledding behind Philips Gym with my eleven-year-old son.) The forces on a snow sledder are gravity, the normal force, sliding friction, and air resistance. Assume that the sliding friction is equal to μ times the normal force, and that the air resistance is proportional to the frontal surface area of the sledder. (a) What is the more important factor in determining who wins a sledding race: mass or frontal area? (b) Assume that a person of height h has a mass proportional to h^3 and a frontal surface area proportional to h^2. Will the best sled racers be tall or short?

84. *Falling bob*
(This problem comes from the version of the Physics Graduate Record Exam administered in 1991–92.) The figure shows a small ball connected to a string, which is attached to a vertical post.



If the ball is released when the string is horizontal, as shown, the magnitude of the ball's acceleration as a function of angle θ is

(a) $g\sin\theta$

(b) $2g\cos\theta$

(c) $2g\sin\theta$

(d) $g\sqrt{3\cos^2\theta+1}$

(e) $g\sqrt{3\sin^2\theta+1}$

85. *It happened on a moonlit night. . .* (HRW section **9**–3, modified from HRW problem 9-16)
Natalie and Ben had been dating seriously for several months when they went out for a canoe trip on a lake. They rented a 65 pound canoe and set off with Ben in the stern and Natalie 9.4 feet forward, near the center. They paddled to a secluded spot near the far shore of the lake, where they stopped to watch the full moon rise over the glass-smooth surface of the lake. After a few minutes they exchanged seats, sharing a quick kiss as they passed one another. Ben noticed that, during this exchange of seats, the bow of the canoe moved 1.2 feet further from the shore. Then they turned the canoe around and paddled back to the dock. Ben knows that he weighs 180 pounds, and now he knows how much Natalie weighs, too. Do you? Comment on any assumptions you (and Ben) must make. (Clue: There is no need to convert from weights to masses, or from English to metric units. Manipulate symbols rather than numbers and you will find at the end of your manipulations that only *ratios* are significant.)

86. *Weighing a flatboat* (HRW section **9**–3)
A 130 pound woman has a stride of 2.4 feet. She stands on a still flatboat floating in still water and takes ten steps towards shore, but she ends up only 18 feet closer to shore. How much does the flatboat weigh?

87. *A girl, a sled, and an ice-covered lake, part II* (HRW section **9**–3)
Additional problem 56 discussed a girl drawing a sled toward her with a rope, when both girl and sled were on a frictionless ice-covered lake. In that problem she pulled with a constant force of 5.2 N, and in part (e) you found the point were the girl and the sled met. Locate the meeting point when the girl instead pulls with (a) a constant force of 3.7 N; (b) a force of 5.2 N for the first half, 3.7 N for the second half; (c) a force that increases linearly until it reaches a maximum of 4.9 N when girl and sled touch.

88. *Diving* (HRW section **9**–3)
In Olympic 10-meter platform diving, about how fast does the diver enter the water? About how much time is available to the diver to perform feats before entering the water? Does your estimate differ for fat vs. skinny divers?

89. *Impulse with force preconceptions* (HRW section **9**–6)
While in truth the net force on a particle results in the acceleration of that particle, many non-physicists hold the naïve preconception that net force results in velocity:

$$\text{correct:} \quad \sum \vec{F} = m\vec{a}$$
$$\text{naïve preconception:} \quad \sum \vec{F} = m\vec{v}.$$

Show that if the naïve preconception were correct, then the net impulse would equal, not the change in momentum, but $m\Delta\vec{r}$.

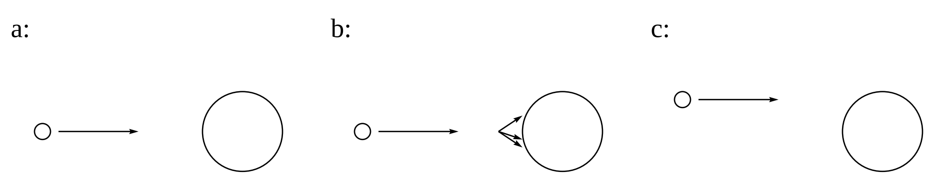
90. *Train latch* (HRW section **9**–9, modified from HRW problem 9-107)
A 48 ton railroad freight car collides with a stationary caboose. The coupling mechanism is activated so the cars latch together and roll down the track attached. During the collision, 34% of the initial kinetic energy dissipates as heat, sound, vibrations, structural damage, and so forth. How much does the caboose weigh?

91. *Asteroid encounter*
Collisions between large astronomical bodies are currently rare in our solar system, but it's thought that they were common during the first few million years of earth's existence. Suppose an asteroid with the mass of the moon (7.35×10^{22} kg) were to smash into the earth (mass 5.976×10^{24} kg) at a speed of 7.3×10^{3} m/s. At what speed would the earth recoil if

a. it were a dead-center collision during which the asteroid buried itself deep into the earth?

b. the asteroid broke up high in the atmosphere and scattered over many square kilometers of terrain?

c. the asteroid struck the edge of the earth and buried itself there?



92. *Sticky skaters* (HRW section **9**–7)
Two 24 kg children, each with a speed of 3.5 m/s, are sliding on a frictionless frozen pond. They collide and stick together because of the Velcro straps on their jackets. The two children then collide with and stick to a 75 kg man who was sliding at 2.1 m/s. After this collision, the three-person composite is stationary. What is the angle between the initial velocity vectors of the two children?

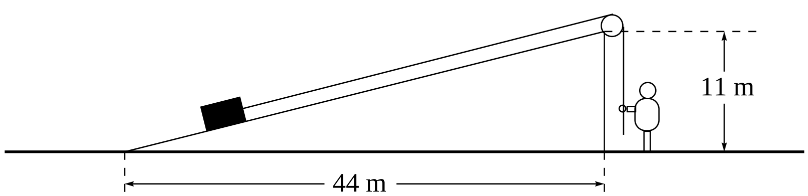
93. *Rolling*

a. In lab we discovered that if a golf ball and a toy car are released from rest and allowed to travel down an inclined ramp, then the car wins (that it, gets to the bottom first). Explain in your own words why this is so.

b. You are designing a toy car to compete in such an inclined ramp race. Everything is finished except for the wheels of the car. By the rules of the competition the wheels cannot exceed a certain weight. What features should you incorporate into your wheel design to make the car go as fast as possible?

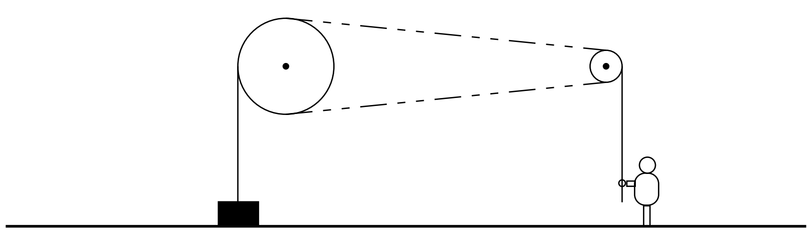
94. *Simple machines*
Adam needs to raise a box weighing 473 N from the ground to a ledge 11 m above the ground. However, he has only the strength to exert a force of 400 N or less.

a. One solution is to construct a ramp with a pulley at the end as shown.



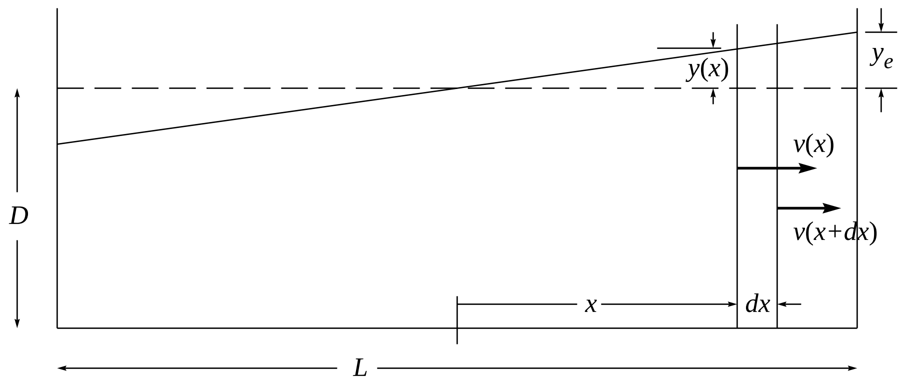
If the ramp and pulley are frictionless, how much work must Adam do to just get the box onto the ledge?

b. A different solution is to pull on a rope wrapped around a 12 inch pulley that is linked via a chain to a 36 inch pulley that has another rope wrapped around it which leads down to the box.



What is the minimum force needed to lift the box off the ground?

95. *Seiches on a lake* (Harmonic oscillation; HRW section **15**–4)
(Modified from a problem in A.P. French, *Vibrations and Waves*, page 74.) This problem is more ambitious than most, in that you must put together a large number of individual pieces to produce the final result. But if you tackle the pieces as suggested, you will find that each piece, by itself, is not particularly difficult, and that the pieces fall together to produce an unexpectedly beautiful and powerful result. (If you cannot solve any of he particular pieces of the puzzle, then assume the stated result and go ahead to put the pieces together.)



You are surely familiar with water sloshing in a bathtub. (If you aren't now, you will be when you start giving baths to your children.) The simplest motion is one in which, to some approximation, the water surface tilts as suggested in the figure but remains more or less flat. A similar phenomenon occurs in lakes and is called a seiche (pronounced "saysh" — see J. Walker, *Flying Circus of Physics*, page 90). Seiches usually start up when dramatic weather fronts produce higher atmospheric pressure at one end of a lake than the other, but the oscillations can continue long after the front has passed.

Consider a lake of rectangular cross section, as shown, with length L, width W, and water depth D. When seiche oscillations occur, the potential energy is due to the slight vertical tilt of the water, while the kinetic energy is almost entirely due to horizontal flow. This problem calculates the period of oscillation.

a. Imagine that at some instant the water level at the edges is $\pm y_e$ with respect to the normal level. (It is the quantity y_e that will execute harmonic oscillation.) Show that at this instant the water surface satisfies

$$y(x) = \frac{2y_e}{L}x,$$

where x ranges from $-L/2$ to $+L/2$.

b. Show that the increased gravitational potential energy of a slab of water between x and $x + dx$ is

$$\tfrac{1}{2}\rho g y^2(x) W\, dx,$$

where ρ is the density of water.

c. Integrate to find that the increased gravitational potential energy of the entire lake is

$$U = \tfrac{1}{6}\rho g L W y_e^2.$$

d. Now we move on to find the kinetic energy, for which we must first find the speed of the water flow. If $D \ll L$, then this flow will be predominantly horizontal. In this case the flow speed v must vary with x, being greatest at the center and falling to zero at the edges. Because water is (to a good approximation) incompressible, the difference in the flow speed at x and the flow speed at $x + dx$ is related to the change dy/dt of the height of the water surface at x, as follows: Water volume flows into the slab at x at the rate vDW and flows out of the slab at $x + dx$ at the rate $(v + dv)DW$. (Assume $y_e \ll D$.) This change in water volume must result in a change of the water height: the rate of change of volume is $(W\,dx)(dy/dt)$. Conclude that v varies with x according to

$$\frac{dv}{dx} = -\frac{1}{D}\frac{dy}{dt} = -\frac{2x}{DL}\frac{dy_e}{dt}.$$

e. Integrate this equation with respect to x to find that

$$v(x) = -\frac{1}{DL}\frac{dy_e}{dt}x^2 + \text{constant},$$

and then fit the boundary conditions $v(-L/2) = v(+L/2) = 0$ to show that

$$v(x) = -\frac{1}{DL}\frac{dy_e}{dt}\left(x^2 - (L/2)^2\right).$$

The negative sign out front gives me pause... when dy_e/dt is positive, I expect that v at any point x will be positive. Does this equation meet that expectation?

f. Show that at any given instant, the kinetic energy due to the horizontal flow of water within a given slab is

$$\tfrac{1}{2}\rho DW(dx)v^2(x)$$

(assuming $y_e \ll D$). Hence integrate over x to find a total kinetic energy of

$$K = \frac{1}{60}\frac{\rho W L^3}{D}\left(\frac{dy_e}{dt}\right)^2.$$

g. All the steps so far have applied to a single instant — the instant sketched in the snapshot. Now we put the system into motion by setting

$$K + U = \text{constant}$$

and comparing to the mass-on-a-spring energy equation

$$\frac{1}{2}m\left(\frac{dy_e}{dt}\right)^2 + \frac{1}{2}ky_e^2 = \text{constant}.$$

The latter equation results in harmonic oscillation with period $2\pi\sqrt{m/k}$. What, then, is the period of harmonic oscillation associated with the former equation?

h. Your resulting equation for the period should depend upon the dimensions D and L, and the acceleration of gravity g, but be independent of the density ρ. Does it make sense that a bathtub of water and a bathtub of mercury would oscillate with the same period?

i. Our model was built with many assumptions: a lake of constant depth, water flow that is purely horizontal, and a flat water surface (actually it is a piece of a sine curve). Nevertheless it provides a picture worthy of comparison to experiment. Lake Geneva, Switzerland, is about 70 km long and has a mean depth of 150 m. The period of its seiche has been measured at 73 min. Compare this with your formula's prediction.

j. (Optional.) Can you understand, in a qualitative way, why the period should be independent of the width W? Why it should increase with increasing length L? Why it should decrease with increasing depth D? Why the period should be longer on the Moon than on Earth? I'm looking for an answer that involves some understanding of the physics of sloshing water, not an answer like "It comes out of the mathematics." (I confess without shame that I cannot find answers to the last three questions. I can make predictions about this situation only through the use of mathematics, not through verbal reasoning. For this reason, I always feel sorry for those in disciplines like English literature or public policy who face questions far more complicated than that of water sloshing in a tub, and yet who must analyze these questions without the benefit of that most powerful and helpful tool: mathematics.)

Chapter 4

Workshops in Classical Mechanics

4.1 Warm Up Problems

The Radius of the Earth

The view from atop Mount Holyoke in Massachusetts (elevation 878 feet) is justly famous. Thomas Cole painted this summit view in his 1836 landscape "The Oxbow," which hangs today in New York's Metropolitan Museum of Art. (The reproduction below comes from http://www.abcgallery.com/C/cole/cole15.html.) Whereas Glacier Point in Yosemite National Park provides a famous a wilderness view; Old Rag in Shenandoah National Park provides a famous pastoral view; and the Empire State Building in New York provides a famous urban view; the view from Mount Holyoke mixes all three of these elements in a sublime way that shows humanity and nature in harmony, rather than in conflict. Indeed, on an exceptionally clear day from the summit of Mount Holyoke you can see the skyscrapers of Hartford, Connecticut — 37 miles away — etched against the horizon.

What is the radius of the Earth?

Death in the City

In the September 11, 2001 World Trade Center attacks, 2,752 were killed. (See "A new account of Sept. 11 loss," New York *Times*, 29 October 2003.)

Estimate the number of individuals who die in New York City in a typical day. (Make reasonable guesses for any demographic data you need.)

The Passenger Pigeon

In connection with the extinction of the Passenger Pigeon, the book *Panati's Extraordinary Endings of Practically Everything and Everybody* (by Charles Panati, Harper & Row, 1989), describes a pigeon hunt held in northern Michigan in 1878: "The pigeons roosted in a compact flock about five miles long by a mile wide. The number of birds was estimated at one billion. Hundreds of local hunters with shotguns began shooting... It took the hunters thirty days to fell the entire pigeon population... A total of three hundred tons of bird went to market."

- A billion pigeons in five square miles implies how many birds per square foot?
- If a billion pigeons were killed in thirty days by hundreds of hunters, how many pigeons did each hunter bag per minute?
- Estimate the weight of one billion dead pigeons, and compare to three hundred tons.
- Comment on the accuracy of this account.

Preparing for War

The administration of G.W. Bush did not present Congress with a budget estimating the costs of the invasion and occupation of Iraq, arguing that because of the vagaries of warfare it was impossible to make any prediction.

Douglas Feith, Undersecretary of Defense for Policy, put it this way:[1] "We're not comfortable with predictions. It is one of the big strategic premises of the work that we do. [Defense Secretary Rumsfeld] is death to predictions. His big strategic theme is uncertainty, the need to deal strategically with uncertainty. The inability to predict the future. The limits on our knowledge and the limits on our intelligence... Nobody will find a single piece of paper that says, 'Mr. Secretary or Mr. President, let us tell you what postwar Iraq is going to look like, and here is what we need plans for.' If you tried that, you would get thrown out of Rumsfeld's office so fast–if you ever went in there and said, 'Let me tell you what something's going to look like in the future,' you wouldn't get to your next sentence!"

[1] "Blind Into Baghdad" by James Fallows (The Atlantic Monthly, January/February 2004).

Yet is the future all that unpredictable? Present an argument that the invasion and occupation of Iraq would cost more than \$300, and less than \$300 trillion. (Presumably Rumsfeld isn't "death to predictions" but "death to accurate predictions".)

More Warm Ups

1. How many hairs are on your head?
2. How many grains of sand are in a one-quart (or one-liter) sand bucket?
3. If all the land on Earth were divided equally among all the people on Earth, how large would your portion be? Give your answer in terms of the size of a typical Oberlin house lot.
4. A hurricane is approaching New Orleans. How many city busses are needed to evacuate 100,000 people? How long will it take to load those people on the busses?
5. Once those 100,000 people are evacuated to high ground, they will need to be supplied with bottled water for drinking. How many semi-truck loads are needed to provide a week's worth of drinking water?
6. Estimate how much time the American public spends each week talking on telephones. If the government were to monitor 1% of these calls, how many analysts (working 40 hours/week) would be needed to listen to them? What would be the total salary of all of these analysts?
7. As of August 2010 the U.S. national debt was about \$13 trillion. What was your share? If the debt were paid off at a rate of \$1 per second, how many years would it take to pay off? This "gee-whiz" way of looking at the debt which is of little practical value, because payments of \$1 per second are very slow compared to typical government payments. A more useful picture emerges by comparing the debt to the total federal government budget, which was about \$3.5 trillion in 2009. If the government operated with a 1% surplus, and applied that surplus to paying off the debt, how long would it take to get it paid off?

4.2 Conclusions

Here are some tips on problem solving drawn through working the warm up problems of the previous section:

Tips obtained through working "The Radius of the Earth"

- First step: Draw a sketch.
- Solve the problem with letter variables: only at the final stage should you plug in numbers.
- Use mnemonic variable names like R for radius of the Earth, h for the height of Mount Holyoke, d for the distance to the horizon. Use these variable names to label your sketch.
- Use the excellent approximation $h \ll R$. Are you making any other assumptions or approximations?
- First find an equation for d as a function of h and R. Is your equation dimensionally consistent?
- It makes sense that d would increase if h increased. Is this what your equation predicts?
- On which planet is the horizon farthest away: the Earth, Mars, or Jupiter? Does your equation agree?
- If both R and h were tripled, then d should triple as well. Does your equation reflect this expectation?
- When you plug in numbers, you will need to find the radius in either feet or miles. Be sure to apply the proper conversions.
- You *could* convert all measurements to, say, kilometers. But in fact SI units are no more "scientific" than any other unit. (The preferred energy unit in atomic physics is 1.60×10^{-19} Joule.)
- Is your result for the radius reasonable? (E.g., a radius of seven miles is *not* reasonable.) You can check using a reference book.
- Use significant figures.

Tips obtained through working "Death in the City" and "The Passenger Pigeon"

- You can't say that a number is "big" or "small" except in relation to another number with the same units.
- "Back of the envelope" approximations are a valuable tool in physics and in daily life. Such approximations give quick approximate answers to problems when high-precision answers aren't needed or would require considerable effort to find.
- You can impress your friends with your skill at drawing significant unexpected conclusions from everyday facts. (However, if you try this too often you might run out of friends.)

4.3 Scaling Problems

Atomic clock

The atomic fountain clock at the National Institute of Standards and Technology (NIST) in Boulder, Colorado sets the time base for the United States. Its goal is to realize the definition of the second as accurately as possible. The second is defined to be 9,192,631,770 oscillations of the radiation emitted from a ^{133}Cs atom at the temperature of zero kelvin. However, the clock operates at room temperature (297 K), not at zero kelvin, and it is known that the oscillation period shifts with temperature. This shift increases as the fourth power of temperature, that is, as T^4. (As a result, the measured period must be corrected before use.) At room temperature, this correction results in a fractional shift of 36.2×10^{-15}. Workers at NIST are currently developing a new clock. How much must they lower the clock temperature of the clock if they want to reduce the shift correction by a factor of 100?

Life expectancy

Biologist have discovered that most species of mammals live for about 1.5 billion heartbeats. They have also discovered that the length of a heartbeat depends on the animal's mass m scaling as $m^{1/4}$.
A mouse lives for about 3 years. If an elephant is 200,000 times more massive than a mouse, how long do elephants live?

4.4 Significant Figures

The impossibility of certainty

No measurement is absolutely certain. For example, in the course of building a tree house I measured a plank with a meter stick and found it to be 187.6 cm long. But a more accurate measurement would of course provide a more accurate length: perhaps 187.64722031 cm. I don't know the plank's exact length, I only know an approximate value. In a math class, 187.6 cm means the same as 187.60000000 cm. But in a physics class, 187.6 cm means the same as 187.6??????? cm, where the question marks represent not zeros, but digits that you don't know. The digits that you *do* know are called "significant digits" or "significant figures."

This has an important philosophical consequence: Because all scientific conclusions are based on measurements, and all measurements contain some uncertainty, no scientific conclusion can be absolutely certain. Anyone worshiping at the alter of science is fooling himself.

This course is more interested in the day-to-day practicalities of uncertainty than in the grand philosophical consequences. How do we express our lack of certainty? How do we work with (add, subtract, multiply, take logarithms of, and so forth) quantities that aren't certain?

Expressing uncertainty

The rule for expressing uncertain quantities is simple: Any digit written down is a significant digit. A plank measured to the nearest millimeter has a length expressed as, say, 103.7 cm or 91.5 cm or 135.0 cm. Note particularly the trailing zero in 135.0 cm: this final digit is significant. The quantity "135.0 cm" is different from "135 cm". The former means "135.0??????? cm", the latter means "135.???????? cm". In the former, the digit in the tenths place is 0, while in the latter, the digit in the tenths place is unknown.

This rule gives rise to a problem for representing large numbers. Suppose the distance between two stakes is 45.6 meters. What is this distance expressed in centimeters? The answer 4560 cm is unsatisfactory, because the trailing zero is not significant and so, according to our rule, should not be written down. This quandary is resolved using exponential notation: 45.6 meters is the same as 4.56×10^3 cm. (This is one of the world's most widely violated rules.)

Working with uncertainty

Addition and subtraction. I measured a plank with a meter stick and found it to be 187.6 cm long. Then I measured a dowel with a micrometer and found it to be 2.3405 cm in diameter. If I place the dowel next to the plank, how long is the dowel plus plank assembly? You might be tempted to say

```
 187.6
+  2.3405
 --------
 189.9405
```

But no! This is treating the unknown digits in 187.6 cm as if they were zeros, when in fact they're question marks. The proper way to perform the sum is

```
 187.6?????
+  2.3405??
 ----------
 189.9?????
```

So the correct answer, with only significant figures written down, is 189.9 cm.

Multiplication and division. The same question mark technique works for multiplication and division, too. For example, a board measuring 124.3 cm by 5.2 cm has an area given through

```
   1243?
  x  52?
--------
   ?????
  2486?
 6215?
--------
 64????
```

Adjusting the decimal point gives an answer of 6.4×10^2 cm^2

But while the question mark technique works, it's very tedious. (It's even more tedious for division.) Fortunately, the following rule of thumb works as well as the question mark technique and is a lot easier to apply:

> When multiplying or dividing two numbers, round the answer down to the number of significant digits in the least certain of the two numbers.

For example, when multiplying a number with four significant digits by a number with two significant digits, the result should be rounded to two significant digits (as in the example above). (If you want a justification of this rule of thumb, you'll have to take Physics 212.)

Evaluating functions. How many significant figures does $\sin(87.2°)$ contain? We know that the real angle is somewhere between $87.200\ldots°$ and $87.300\ldots°$, so the real sine is somewhere between

$$\begin{aligned} \sin(87.200\ldots°) &= 0.9988061\ldots \\ \sin(87.300\ldots°) &= 0.9988898\ldots \end{aligned}$$

The usual rule is to make the last significant digit in the result to be the first one from the left that changes when you repeat the calculation. In this case the first digit that changed was the "0" that changed to an "8", so we round the result to five significant figures, namely

$$\sin(87.2°) = 0.99881.$$

Numbers that are certain

Any measured number is uncertain, but *counted* and *defined* quantities can be certain. If there are seven people in a room, there are 7.0000000... people. There are never 7.00395 people in a room. And the inch is *defined* to be exactly 2.5400000... centimeters — there's no uncertainty in this conversion factor, either.

Conclusions

This course is not going to nitpick about significant figures — for example I won't be upset if you round differently from me and get a result of 130.5 cm where I get a result of 130.3 cm; and I won't carp if you write 4100 miles instead of 4.1×10^3 miles — but I do expect you to use significant figures throughout the course — and I *will* carp if you write 4116.35 miles when you should write 4100 miles. Every problem in this course is worth 10 points, and I'll take off 2 points if you make a significant error in significant figures.

There's a lot more to know about uncertainty, but this is what you'll need for Physics 110.

4.5 Dimensions

What does "dimensions" mean?

Suppose I say that a table is six feet long or, in symbols,

$$\ell_T = 6 \text{ ft},$$

where ℓ_T represents the length of the table. This means that the table is six times as long as the length of the standard foot:

$$\ell_T = 6 \text{ ft} \quad \text{means} \quad \ell_T = 6 \times (\text{the length of the standard foot}).$$

In other words, the symbol "ft" used in the equations above stands for "the length of the standard foot".

The symbol "ℓ_T" stands for "6 ft". That is, it stands for the number "six" times the length of the standard foot, or in other words, for the number "six" times the unit "ft". If I wrote "$\ell_T = 6$" instead of "$\ell_T = 6$ ft", I'd be dead wrong. . . just as wrong as if the solution to an algebra problem were "$y = 6x$" and I wrote "$y = 6$", or if the solution to an arithmetic problem were "6×7" and I wrote "6". In all three cases, my answer would be wrong because it omitted a factor. (These are, respectively, the factor of "the length of the standard foot", the factor of x, and the factor of 7.) The length of the table is not 6 — rather, the ratio of the length of the table to the length of the standard foot is 6.

Ignoring the units of a measurement results in practical as well as conceptual error. On 23 September 1999 the "Mars Climate Orbiter" spaceprobe crashed into the surface of Mars, dashing the hopes and dreams of dozens of scientists and resulting in the waste of $125 million. This spacecraft had survived perfectly the long and perilous trip from Earth to Mars. How could it have failed so spectacularly the final, relatively easy, phase of its journey? The manufacturer, Lockheed Martin Corporation, had told the the spacecraft controllers, at NASA's Jet Propulsion Laboratory, the thrust that the probe's rockets could produce. But the Lockheed engineers gave the thruster performance data in pounds (the English unit of force), *and they didn't specify which units they used.* The NASA controllers assumed that the data were in newtons (the metric unit of force). (New York *Times*, 1 October 1999, page 1. This is an embarrassing place to have your blunders reported.)

Two Teams, Two Measures Equaled One Lost Spacecraft

By ANDREW POLLACK

LOS ANGELES, Sept. 30 — Simple confusion over whether measurements were metric or not led to the loss of a $125 million spacecraft last week as it approached Mars, the National Aeronautics and Space Administration said today.

An internal review team at NASA's Jet Propulsion Laboratory said in a preliminary conclusion that engineers at Lockheed Martin Corporation, which had built the spacecraft, specified certain measurements about the spacecraft's thrust in pounds, an English unit, but that NASA scientists thought the information was in the metric measurement of newtons.

The resulting miscalculation, undetected for months as the craft was designed, built and launched, meant the craft, the Mars Climate Orbiter, was off course by about 60 miles as it approached Mars.

"This is going to be the cautionary tale that is going to be embedded into introductions to the metric system in elementary school and high school and college physics till the end of time," said John Pike, director of space policy at the Federation of American Scientists in Washington.

Lockheed's reaction was equally blunt.

"The reaction is disbelief," said Noel Hinners, vice president for flight systems at Lockheed Martin Astronautics in Denver, Colo. "It can't be something that simple that could cause this to happen."

The finding was a major embarrassment for NASA, which said it was investigating how such a basic error could have gone through a mission's checks and balances.

"The real issue is not that the data was wrong," said Edward C. Stone, the director of the Jet Propulsion Laboratory in Pasadena, Calif., which was in charge of the mission. "The real issue is that our process

Continued on Page A16

Modern information technology actually encourages mistakes like this. When you use a calculator, a spreadsheet, or a computer program, you enter pure numbers like "1.79", rather than quantities like "1.79 feet". So it's especially important to be on your guard and document your units when using computers. Keep the units in your mind, even if you can't keep them in your calculator!

A nitpicky distinction is that the length of the table has the *units* of "feet" and the *dimensions* of "length". If the length of the table were measured in yards or meters it would still have the dimensions of length. But in everyday language the terms "units" and "dimensions" are often used interchangeably.

Unit conversions

It is standard usage to refer to the length of the standard foot by the symbol "ft". But in this document I'll also refer to the length of the standard foot by the symbol ℓ_F. Similarly I'll call the length of the standard yard either "yd" or ℓ_Y.

You know that if a table is 6 feet long it is also 2 yards long:

$$\ell_T = 6 \text{ ft} = 6\ell_F = 2 \text{ yd} = 2\ell_Y.$$

How can this be? It's certainly *not* true that 6 = 2! It's true instead that 6 ft = 2 yd because the length of a yardstick is three times the length of a foot ruler:

$$\ell_Y = 3\ell_F.$$

This tells us that

$$2 \text{ yd} = 2\ell_Y = 2 \times (3\ell_F) = 6\ell_F = 6 \text{ ft},$$

or, in the opposite direction,

$$6 \text{ ft} = 6\ell_F = 6\ell_F(1) = 6\ell_F\left(\frac{\ell_Y}{3\ell_F}\right) = 6\cancel{\ell_F}\left(\frac{\ell_Y}{3\cancel{\ell_F}}\right) = 2\ell_Y = 2 \text{ yd}.$$

In short, the symbol "ft" can be manipulated exactly like the symbol "ℓ_F", because that's exactly what it means!

Incompatible dimensions

If I walk for 4 yd, and then for 2 ft, how far did I go? The answer is 14 ft or $4\frac{2}{3}$ yd, but not $4 + 2 = 6$ of anything!

If I walk for 4 yd, and then pause 30 seconds, how far did I go? Certainly *not* 4 yd + 30 sec. The number 34 has no significance in this problem. For example, it can't be converted into minutes.

In general, *you can't add quantities with different units.*

This rule can be quite helpful. Suppose you're working a problem that involves a speed v and a time t, and you're asked to find a distance d. Someone approaches you and whispers: "Here's a hint: use the equation $d = vt + \frac{1}{2}vt^2$." You know that this hint is wrong: The quantity vt has the dimensions of [distance], but the quantity $\frac{1}{2}vt^2$ has the dimensions of [distance×time]. You can't add a quantity with the dimensions of [distance×time] to a quantity with the dimensions of [distance], any more than you could add 30 sec to 4 yd.

Dimensions aren't a cure-all

It's not meaningful to add quantities with different units, but just because two quantities do have the same units doesn't mean it must be meaningful to add them. For example, two quantities that you will encounter in your mechanics course are work and torque. Both have the dimensions of [force×distance], but the entities are quite distinct and it never makes sense to add a quantity of work to a quantity of torque. Why? Work is defined as a force times a distance parallel to that force, while torque is defined as a force times a distance perpendicular to that force. They are different types of entity, even though they share the same dimensions.

Another example: The rate at which a conveyer belt delivers gravel is measured in kilograms/second. And if a frictional force is proportional to velocity, $F_{\text{friction}} = -bv$, then (as we will soon see) the friction coefficient b has the units of kilograms/second. But these are clearly entities of different character!

A famous use of dimensional analysis

The following story about the use of dimensional analysis comes from David L. Goodstein, *States of Matter* (Prentice-Hall, 1975, page 436). (See also *Physics Today*, May 2000, page 35; and G.I. Barenblatt, *Scaling Phenomena in Fluid Mechanics* (Cambridge University Press, 1994).)

"Dimensional analysis is a technique by means of which it is possible to learn a great deal about very complicated situations if you can put your finger on the essential features of the problem. An example is the well-known story of how G.I. Taylor was able to deduce the yield of the first nuclear explosion from a series of photographs of the expanding fireball in *Life* magazine. He realized that he was seeing a strong shock expanding into an undisturbed medium. The pictures gave him the radius as a function of time, $r(t)$. All that could be important in determining $r(t)$ was the initial energy release, E, and the density of the undisturbed medium, ρ. The radius, with the dimension of length, depended on E, ρ, and t, and he constructed a distance out of these quantities. E and ρ had to come in as E/ρ to cancel the mass. E/ρ has the dimensions [length]^5/[time]^2, so the only possible combination was

$$r(t) \propto \left(\frac{E}{\rho} t^2\right)^{1/5}.$$

A log-log plot of r versus t (measured from the pictures) gave a slope of $\frac{2}{5}$, which checked the theory, and E/ρ could be obtained from extrapolation to the value of $\log r$ when $\log t = 0$. Since ρ, the density of undisturbed air, was known, E was determined to within a [dimensionless] factor of order one. For the practitioner of the art of dimensional analysis, the nation's deepest secret had been published in *Life* magazine."

Problem

Sound speed

The speed of sound v_s in air, in a given room, could reasonably depend on three things: the air density ρ, the air pressure p, and the room volume V. In other words

$$v_s = C\rho^x p^y V^z$$

where C is some dimensionless number. Use dimensional analysis to determine the exponents x, y, and z. The SI units for pressure are kg/ms^2.

4.6 Acceleration Problem

A ball bounces up and down. While the ball goes down, it is speeding up ("quickening"); while it goes up, it is slowing down ("slackening"). In addition, when the ball reaches the very top of its trajectory it has zero velocity at an instant (it does *not* hover). We saw last week that these three, seemingly disparate, motions are in fact just three reflections of uniform acceleration at -9.8 m/s^2 (positive direction taken upward). The problem below attempts to make this point in a different way.

Problem

Adam drives due east on Highway 20 at 50 mph. He presses down on the gas petal, and 5 seconds later his speed is 60 mph. Using the standard convention that east is positive, what is Adam's average acceleration (with sign) during this five-second interval? During this interval is Adam "tossed back against his seat" (towards the west) or "tossed forward against his seat belt" (towards the east)?

Jennifer drives due west on Highway 20 at 80 mph. She presses down on the break petal, and 5 seconds later her speed is 70 mph. Using the standard convention that east is positive, what is Jennifer's average acceleration (with sign) during this five-second interval? During this interval is Jennifer "tossed back against her seat" (towards the east) or "tossed forward against her seat belt" (towards the west)?

Solution

Adam's average acceleration is

$$\frac{\text{(final velocity)} - \text{(initial velocity)}}{\text{time elapsed}} = \frac{(60\text{ mph}) - (50\text{ mph})}{5\text{ sec}} = +2\text{ mph/sec.}$$

Adam is "tossed back against his seat", that is, towards the west.

Jennifer's average acceleration is

$$\frac{\text{(final velocity)} - \text{(initial velocity)}}{\text{time elapsed}} = \frac{(-70\text{ mph}) - (-80\text{ mph})}{5\text{ sec}} = +2\text{ mph/sec.}$$

Jennifer is "tossed forward against her seat belt", that is, towards the west.

Moral

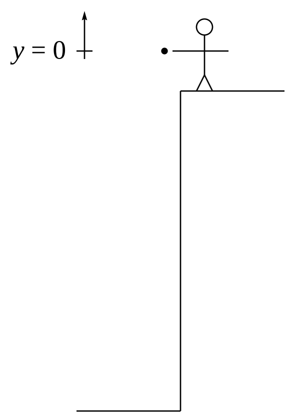
Both drivers undergo the same acceleration and hence both experience the "tossing" in the same direction. However their cars are pointed in different directions so in Adam's case the "toss" is towards the seat while in Jennifer's case it's towards the seat belt.

4.7 Talking About Motion With Constant Acceleration

In the beauty and elegance and power of the calculus, in the cleverness of Galileo's arguments, in the concerns of dimensional analysis and questions of applicability — it is easy to forget that a constant acceleration of -9.8 m/s^2 means nothing more than that the object falls downward 9.8 m/s faster each second.

To explore this, let's work out a few dropping problems using not calculus, not algebra, but just arithmetic. (Calculus and algebra are powerful tools that solve difficult problems with ease. I don't advocate your forgetting calculus and algebra altogether. But the very power of these tools can obscure "what's really going on" with a screen of symbols and equations. A rotary saw is far more powerful than a hand saw, but this power makes it also far more dangerous.) To make the arithmetic easier, we'll use the approximation $g \approx 10$ m/s^2.

Problem 1: You are standing at the top of a cliff and drop a ball. How fast is it going after 0, 1, 2, or 3 seconds?



Solution 1: At time $t = 0$ s, it has velocity 0 m/s. Each second, it picks up an additional 10 m/s in downward velocity. (This is the *meaning* of constant acceleration.) Thus the velocity depends on time as shown in this table:

seconds after release	velocity (m/s)
0	0
1	-10
2	-20
3	-30

Problem 1a: A brick falls from the top of a skyscraper and hits the sidewalk after about 3 seconds. Estimate its speed in miles per hour. (Hint: What is g in the units mph/sec?)

Problem 2: You are standing at the top of a cliff and drop a ball. What is its position after 0, 1, 2, or 3 seconds?

Solution 2: During the first second, the initial velocity is 0 m and the final velocity is -10 m/s, whence the average velocity is -5 m/s. Now

$$v_{\mathrm{ave}} = \frac{\Delta y}{\Delta t} \quad \text{so} \quad \Delta y = v_{\mathrm{ave}} \Delta t. \tag{4.1}$$

So for the first second, $v_{\mathrm{ave}} = -5$ m/s and $\Delta t = 1$ s together imply that $\Delta y = -5$ m. That is, the ball drops 5 meters during the first second.

During the second second, the initial velocity is -10 m/s and the final velocity is -20 m/s — so the average velocity is -15 m/s. Thus

$$\Delta y = v_{\mathrm{ave}} \Delta t = (-15 \text{ m/s})(1 \text{ s}) = -15 \text{ m}. \tag{4.2}$$

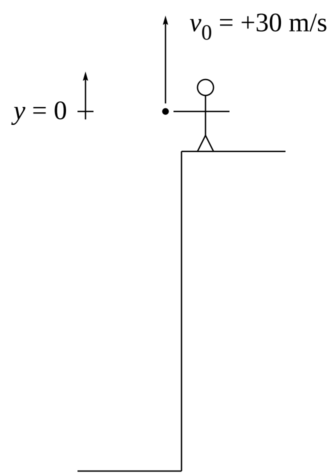
That is, the ball drops 15 meters during the second second, so at $t = 2$ s it is 20 meters below the dropping point.

Similarly for the third second: the average velocity is -25 m/s so the ball drops 25 meters to end up 45 meters below the dropping point.

In summary:

seconds after release	velocity (m/s)	position (m)
0	0	0
1	-10	-5
2	-20	-20
3	-30	-45

Problem 3: You toss a ball upward at 30 m/s from a cliff top. How fast is it going after 0, 1, 2,. . . 7 seconds?



Solution 3: At time $t = 1$ s, it has velocity 30 m/s. Each second, the velocity changes by -10 m/s. (This is the *meaning* of constant acceleration.) Thus the velocity depends on time as shown in this table:

seconds after release	velocity (m/s)
0	+30
1	+20
2	+10
3	0
4	−10
5	−20
6	−30
7	−40

Problem 4: You toss a ball upward at 30 m/s from a cliff top. What is its position after 0, 1, 2,. . . 7 seconds?

Partial solution 4: Use the "average velocity" method of solution 2. I've filled in two of the values in this table. Can you find the rest of them?

seconds after release	velocity (m/s)	position (m)
0	+30	
1	+20	+25
2	+10	
3	0	
4	−10	
5	−20	
6	−30	
7	−40	−35

Problem 5: A helicopter lifts off from the base of the cliff, and flies straight upward at constant velocity 30 m/s. The helicopter passes you at the same instant that you toss the ball upward with velocity 30 m/s. Change your solution for problem 4 to reflect the situation as seen from the helicopter. (That is, show the velocity relative to the helicopter and the position relative to the helicopter.)

Problem 6: Compare your solution to problem 5 with your solution to problem 2.

Conclusions:

There is a strong temptation to switch the reference axis from upward to downward at the top of the flight. Don't do it. . . there is nothing special going on at the top of the flight.

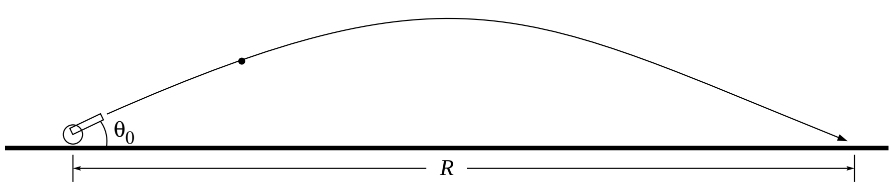
The ball has zero velocity at the top of the flight, but only for an instant. (Just as it has velocity −30 m/s at $t = 6$ s, but only for an instant.)

The helicopter's reference frame is just as good as your reference frame. (Compare the "pouring coffee in a jet plane" argument.)

The word "fall" has one meaning in everyday life, and a different meaning in physics.

4.8 The Range Equation

A cannon aimed at angle θ_0 launches a projectile of mass m at speed v_0. How far does it go before hitting the Earth again?



According to your textbook, this distance, called the "range", is given through the equation

$$R = \frac{v_0^2}{g}\sin(2\theta_0),$$

where g represents the acceleration due to gravity. Your text includes a derivation of this equation, but I want to do more than derive it. . . I want to understand it.

Assumptions

The range equation is not exact — it is derived from a model that ignores effects such as:

the size of the projectile
air friction
curvature of the Earth
height of the cannon
rotation of the Earth
gravitational attraction is weaker when farther from the Earth
relativistic effects

Toss a balloon into the air, and you'll see that it follows a trajectory nothing like the parabola sketched above. Nevertheless, this imperfect range equation is applicable to many situations.

History

This is a problem of obvious importance to baseball players, acrobats, and military planners. Galileo's *Discourses and Demonstrations Concerning Two New Sciences* contains the first systematic treatment of the problem. After the derivation of the range equation, one of the debaters remarks that

> The force of rigid demonstrations such as occur only in mathematics fills me with wonder and delight. From accounts given by gunners, I was already aware of [these facts]. . . but to understand why this happens far outweighs the mere information.

Modern military cannons frequently lob shells over dozens of miles, and the assumptions mentioned above lead to inaccuracies that are not acceptable in military situations. To aim such artillery you must first solve complicated differential equations that take into account issues like air resistance and the rotation of the Earth. During World War II, each model of artillery was shipped with a "range table" that had the differential equations solved beforehand. It took about three months to calculate a range table, and so many new artillery models were coming out that it seemed the war might end before the necessary range tables were produced. To solve this problem, the army turned to J. Presper Eckert at the Moore School of Electrical Engineering at the University of Pennsylvania. Eckert solved this problem by inventing a new device: the electronic computer.

Investing the equation with meaning

Does the range equation make sense?

The dimensions check out. Indeed, this is the *only* way to combine a velocity (v_0) and an acceleration (g) to form a quantity with the dimensions of length (R).

If you increase g, you decrease the range. Does this make sense? Yes: move the experiment to Jupiter, and the projectile won't fly as far — move it to the Moon, and it will fly farther.

If you increase v_0, you increase the range. Does this make sense? Yes: faster leads to farther.

In fact, range goes up like the *square* of v_0. Can we understand why it goes like the square rather than linearly? I think we can. If v_0 increases, the projectile spends more time in the air (one factor of v_0). Furthermore, during its time in the air it's moving faster (second factor of v_0). [If you find this argument unconvincing, I don't blame you. It's suggestive but not definitive.]

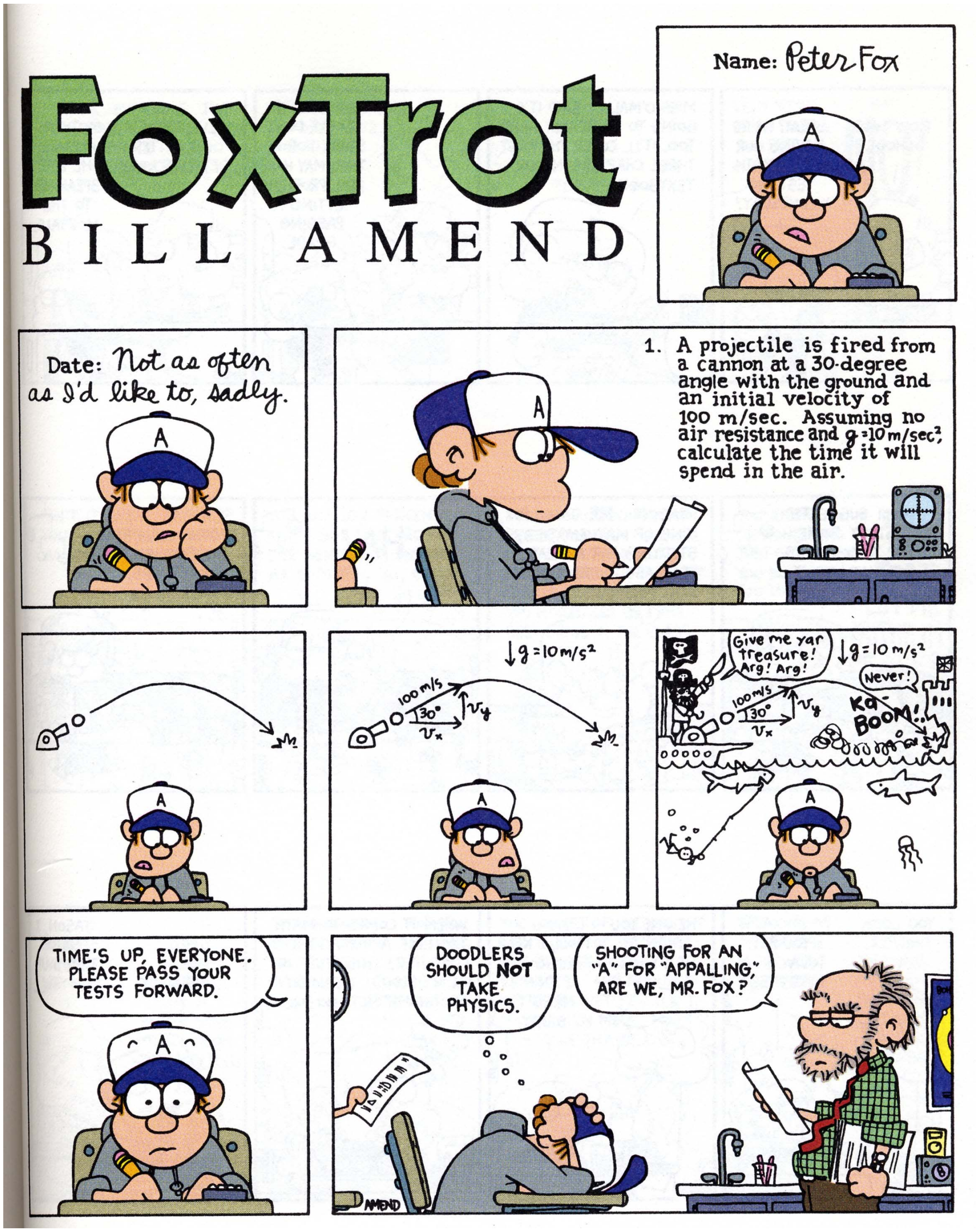
If $g = 0$ the range is infinite. Sure enough, without gravity the cannonball would never fall back to Earth.

If $v_0 = 0$ the range is zero. Yes, it just falls straight down.

Suppose the launch angle θ_0 is varied: When θ_0 is small, the range is small. The range increases until it reaches a maximum when $\theta_0 = 45°$. Then it decreases with trajectories that fly high into the air but then come back to Earth near the launch point. Finally, $\theta_0 = 90°$ gives a trajectory that goes straight up and then straight back down for a resulting range of zero.

What is missing from the equation? The range doesn't depend on the color of the projectile, or its size (assuming that air friction is negligible), or the roughness of its surface. All of these are reasonable. But one missing item is not so obvious: The range doesn't depend on the projectile's *mass*. This is the same surprise that all objects fall with the same acceleration, $-9.8\ \mathrm{m/s^2}$, near the surface of the Earth (making the usual assumptions). Galileo had a cute argument about this surprise, which we discussed in class and reproduced in section 2.5. Still, to me at least, although we've cataloged and discussed this surprise, it's still a surprise.

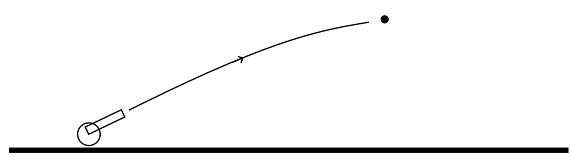
4.9 Cannonball Flight Time



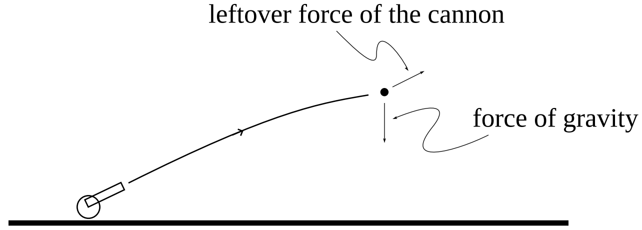
The problem in the above cartoon can be solved in at least two different ways. (My son in seventh grade solved it one way — although he didn't know trigonometry and so had to be told that the vertical component of the initial velocity was 50 m/s.)

4.10 "Leftover Force"

I was once chatting with an Oberlin College psychology professor about the teaching of force and motion. I sketched the situation below: A cannon launches a cannonball into the air.



I asked the psychology professor to sketch the forces acting on the cannonball. She drew an arrow pointing downward, and labeled it "force of gravity." So far so good. But then she drew an arrow parallel to the cannon, and labeled it "leftover force of the cannon."



I told her that, from the point of view of physics, the only force acting on the cannonball was the force of gravity, that the cannon *had* exerted a force on the cannonball back when it was within the bore of the cannon, but now that the ball was flying through the air this force was no longer acting. (That is, no longer pushing or pulling.)

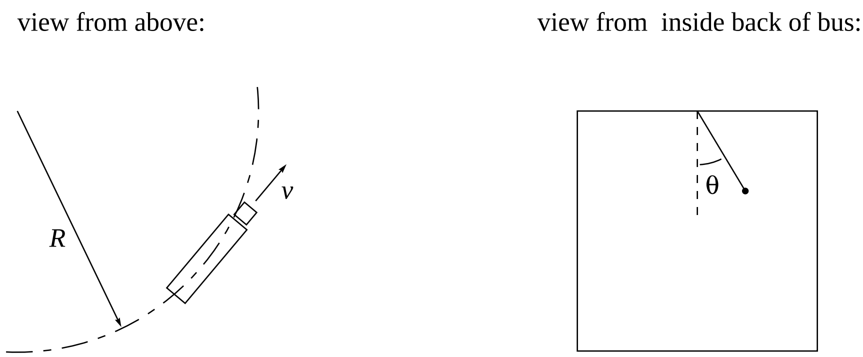
She responded that this was precisely why the force in question was "leftover", and that I might be able to convince her mathematically that the only force acting was the force of gravity, but that I'd never be able to convince her through a verbal argument that there was no such thing as the "leftover force of the cannon."

And indeed, I wasn't able to. I'm so wrapped up in the language and concepts of physics that it's just plain obvious to me that there's no such thing as "leftover force". However, to tell her "it's obvious" would not be a convincing argument. You're less wrapped up in physics than I am... would you please help me out by supplying an argument that might convince the psychology professor?

⟦The idea of "leftover force" is closely related to the medieval concept of "impetus." See Allan Franklin, "Principle of inertia in the Middle Ages," *American Journal of Physics* **44**, 529–545 (1976).⟧

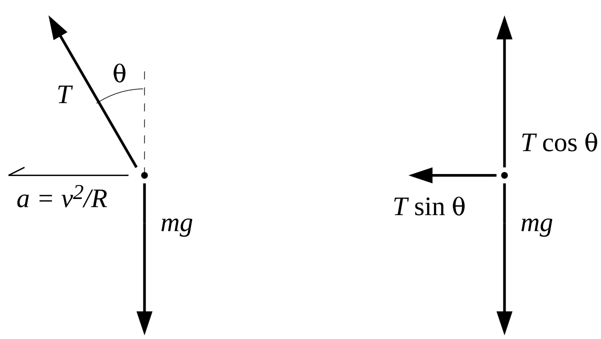
4.11 Life in an Accelerating Vehicle

A ball hangs from the ceiling of a bus on a cord. The bus, moving at constant speed v, takes a left turn along a circular path of radius R.



Inside the bus, the ball hangs at angle θ. What is θ in terms of v, R, and g?

Draw a free body diagram showing the forces on the ball, and the acceleration of the ball. On the right, the tension is resolved into horizontal and vertical components.



Apply $\sum \vec{F} = m\vec{a}$ to first the horizontal, then the vertical components:

$$\sum F_x = ma_x \implies -T\sin\theta = -m\frac{v^2}{R}$$
$$\sum F_y = ma_y \implies T\cos\theta - mg = 0.$$

Simplify to find:

$$\begin{aligned} T\sin\theta &= m\frac{v^2}{R} \\ T\cos\theta &= mg. \end{aligned}$$

Divide the first equation by the second to find

$$\tan\theta = \frac{v^2}{Rg}.$$

(Note that T and m cancel out!)

There is no force "holding the ball to the right." (You don't need a force to keep an object stationary.) Instead, the ball is accelerating to the left... the ball is "tossed to the right" so that the horizontal component of the tension will supply the force necessary for the acceleration to the left.

More examples:

A car accelerates forward... the passengers are "tossed backward" so that the seat backs can supply the needed forward force. (Just as depressing a pillow creates a force.)

(This is why I use the term "tossed backward" rather than "pushed backward". The passengers accelerate forward, so the force – the push – is forward not backward.)

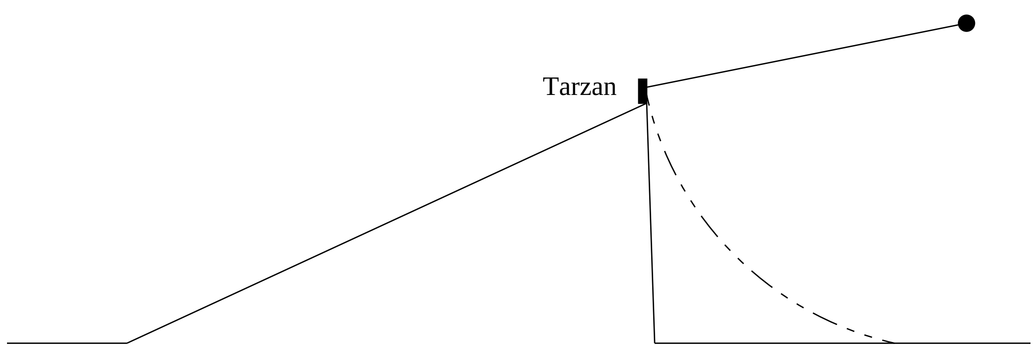
A car breaks... the passengers are "tossed forward" so that the seat belts can supply the needed backward force.

A car turns left... the passengers are "tossed to the right" so that the car walls can supply the needed leftward force.

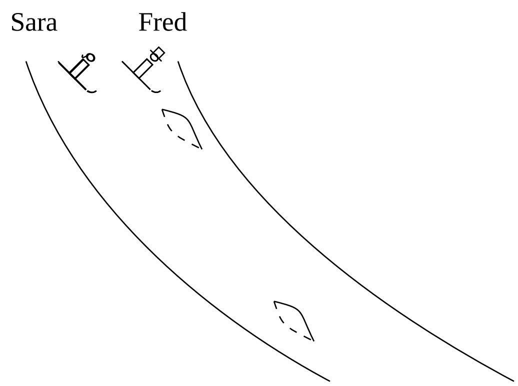
4.12 Assorted Problems

1. Bridge. A bridge spans a placid lake at 5.3 m above the water surface. Someone standing on the bridge wants to toss a rock at 7.2 m/s, and he wants it to hit the water with the greatest possible speed. At what angle should he toss the rock? (Air resistance is negligible.) (*Clue:* Why is it easy to find the angle giving the maximum speed but hard to find the angle giving the maximum distance?)

2. Tarzan. Tarzan is standing at the top of a mountain and needs to get to its base. He has the choice of (1) jumping off the steep side, (2) skiing down the shallow side, or (3) swinging down on a vine. In all three cases, friction is negligible. With which method will he have the highest speed when he reaches the base? Which method will take longest?



3. Skiing. Two skiers race down the same slippery slope, but Fred encounters a small hillock of snow near the top of the slope and Sara encounters a similar small hillock of snow near the bottom of the slope. Ignore friction.



Student A argues: “According to the conservation of energy, Fred and Sara will be traveling at the same speed when they reach the bottom of the slope. The two skiers travel the same distance. (They both have “extra” distance to travel because of the hillocks, but each has the same extra distance.) Therefore both skiers will take the same amount of time to descend.” Student B argues: “You’re *almost* right, but in fact Fred encounters his hillock (and its extra distance) when he’s going slow, whereas Sara encounters her hillock (and its extra distance) when she’s going fast, so Sara will come in before Fred.” Student C argues: “You’re both wrong. Sara encounters her hillock at the critical moment just before the end of the race, so Sara will come in last.” Which student is correct?

4.13 The Height of Heaven

In *Paradise Lost*, Milton describes the fall of Satan from heaven to earth:

> ...from Morn
> To Noon he fell, from Noon to dewy Eve,
> A Summer's day; and with the setting Sun
> Dropt from the Zenith like a falling Star ...

It is clear that air resistance can be ignored in this trip, which was mostly through outer space. Assume that Satan starts from rest and falls for a full "Summer's day" (17.5 hours of daylight), and calculate the height of heaven. (Remember that the acceleration due to gravity is smaller up near heaven than it is on the surface of the earth.)

Use:

m for mass of Satan (unknown)
M_E for mass of earth (known)
r_H for distance from center of earth to heaven (desired)
r_E for radius of earth (known)
G for gravitational constant (known)
T for time required to fall (known: 17.5 hour = 6.30×10^4 s.)

The only force acting on Satan during this fall is the conservative force of gravity, with potential energy function

$$-G\frac{mM_E}{r}.$$

Energy conservation tells us that when Satan reaches height r above the center of the earth, his speed v is given through

$$-\frac{GM_Em}{r_H} = \tfrac{1}{2}mv^2 - \frac{GM_Em}{r}.$$

Fortunately, m cancels from all sides of this equation: otherwise we wouldn't be able to solve the problem. The speed is given through

$$v^2 = \left(\frac{dr}{dt}\right)^2 = 2GM_E\left(\frac{1}{r} - \frac{1}{r_H}\right)$$

or

$$\frac{dr}{dt} = -\sqrt{2GM_E}\sqrt{\left(\frac{1}{r} - \frac{1}{r_H}\right)}.$$

(I take the negative sign of the square root because Satan is going down, so dr/dt is negative.)

This is a differential equation that we can solve for the function $r(t)$. Write

$$\frac{dr}{\sqrt{1/r - 1/r_H}} = -\sqrt{2GM_E}\,dt$$

and integrate both sides to find

$$\int_{r_H}^{r_E} \frac{dr}{\sqrt{1/r - 1/r_H}} = -\sqrt{2GM_E}\int_0^T dt$$

or

$$\int_{r_E}^{r_H} \frac{dr}{\sqrt{1/r - 1/r_H}} = \sqrt{2GM_E}\,T.$$

The only thing standing between us and victory is that one integral. I'll start by writing $x = r/r_H$, so that x is a dimensionless variable (throughout the fall, $1 \geq x > 0$):

$$\int_{r_E/r_H}^{1} \frac{r_H\,dx}{\sqrt{(1/xr_H) - (1/r_H)}} = \sqrt{r_H^3}\int_{r_E/r_H}^{1} \frac{dx}{\sqrt{(1/x) - 1}} = \sqrt{2GM_E}\,T.$$

and then

$$\int_{r_E/r_H}^{1} \frac{dx}{\sqrt{(1/x) - 1}} = \sqrt{\frac{2GM_E T^2}{r_H^3}}.$$

This integral is best approached through substitution. Use

$$u = \frac{1}{x} \quad \text{so that} \quad du = -\frac{1}{x^2}\,dx = -u^2\,dx$$

(throughout the fall, $u \geq 1$) to find

$$\int_{r_E/r_H}^{1} \frac{dx}{\sqrt{(1/x) - 1}} = -\int_{r_H/r_E}^{1} \frac{du}{u^2\sqrt{u-1}}.$$

This integral is tabulated in H.B. Dwight's *Tables of Integrals*, equation 192.21, as

$$-\int_{r_H/r_E}^{1} \frac{du}{u^2\sqrt{u-1}} = -\left[\frac{\sqrt{u-1}}{u} + \arctan\sqrt{u-1}\right]_{r_H/r_E}^{1}.$$

(The function $\arctan(x)$ in radians.) So

$$-\int_{r_H/r_E}^{1} \frac{du}{u^2\sqrt{u-1}} = +\left[\frac{\sqrt{r_H/r_E - 1}}{r_H/r_E} + \arctan\sqrt{r_H/r_E - 1}\right],$$

and the exact solution to our problem comes from solving

$$\frac{\sqrt{r_H/r_E - 1}}{r_H/r_E} + \arctan\sqrt{r_H/r_E - 1} = \sqrt{\frac{2GM_E T^2}{r_H^3}}$$

for the unknown r_H.

The easiest way to do this is to define the ratio $u_f = r_H/r_E$ (u_f stands for "u final"), so that the height of heaven is

$$\text{height of heaven } = r_H - r_E = (u_f - 1)r_E.$$

We seek the value of u_f such that

$$\frac{\sqrt{u_f - 1}}{u_f} + \arctan\sqrt{u_f - 1} = \sqrt{\frac{2GM_E T^2}{u_f^3 r_E^3}}$$

or

$$\sqrt{u_f(u_f-1)} + \sqrt{u_f^3}\arctan\sqrt{u_f-1} - \sqrt{\frac{2GM_ET^2}{r_E^3}} = 0.$$

[To three significant figures, the dimensionless quantity above is

$$\sqrt{\frac{2GM_ET^2}{r_E^3}} = 111.]$$

This equation has no analytic solution, but it's easy to solve numerically. I used my trusty old HP-15C calculator to find the solution

$$u_f = 17.1 \quad \text{whence: height of heaven } = (u_f - 1)r_E = 1.03\times 10^8 \text{ m.}$$

So heaven's not that far away. . . it's about one-quarter the distance to the moon (namely 3.82×10^8 m). (You might recall that it took the Apollo astronauts about three days to fall from the moon to the earth.)

4.14 Tips for the Second Exam

Physics Tips:

- **Don't leave blank.** You'll get *some* points for doing as little as sketching the situation, or writing "This is a work-energy problem but I don't know how to solve it." If you can do only part of a problem (perhaps even the last part) then do it! This advice has been known since biblical times as "Don't hide your light under a bushel."

- **Show your reasoning.** Suppose you give only the final answer, whether a number or an equation. If it's right, then you've earned partial credit. But if it's wrong, then I can't award you any credit at all.

- **No need to convert units.** There is a misconception that all science is done in SI units. (And a related misconception that all work done in SI units is scientific. . . this is why "research workers" in ESP always report their "results" in SI units.) But in fact, different sciences have their own special units. In astronomy, masses are usually reported not in grams or in kilograms but in terms of the mass of our sun: Sirius is said to have a mass of about 2.35 solar masses. Energies in atomic physics are usually reported in term of the "electron volt," which is a unit of energy equal to 1.6×10^{-19} J. You need to use a consistent set of units (don't add 2 feet plus 6 inches to find 8 of anything) but that consistent set of units might be the SI system or any other consistent set.

- **State your units.** "The mass is 5.3." is not an answer. Is that 5.3 grams, 5.3 kilograms, or 5.3 solar masses? There's no need to convert units, but if you do, then you must state what you're converting to.

- **What is force?** In the physics meaning of the word "force", a force acts at an instant. When a bullet flies through the air, the only forces acting upon it are gravity and air resistance. At some time in the past it was subjected to the force from a powder explosion within a gun, and this force gave the bullet whatever velocity it had as it left the barrel. But once it leaves the barrel that force no longer acts, and is irrelevant to its future motion.

- **Starting up a problem.** Are you faced with a problem and you don't know where to start? Then: (1) Sketch the situation. Sketching often gives you a hint as to where to go next. (2) Classify the problem. Is this a kinematics problem, a force problem, a momentum problem, an energy problem? Nearly every problem we've seen in this course falls into one of these four categories.

- **Force problems.** Draw one diagram to show the geometry, and then a separate free body diagram for each object of interest. Each free body diagram must show only the forces *acting on* that object, not the forces *exerted by* that object. In this course, each force will be due either to gravity or to contact with another object. The sum of those forces is $m\vec{a}$: that is, the acceleration is a result of the sum of those gravitational and contact forces. The quantity $m\vec{a}$ is not itself a force and must not be drawn or treated as one.

- **Energy problems.** If mechanical energy is conserved, then $E_{\text{final}} = E_{\text{initial}}$. In many common situations (such as a ball dropping from rest) this results in kinetic energy (final) equal to potential energy (initial). But in general it is not true that kinetic energy equals potential energy.

General Test-taking Tips:

- Get a good night's **sleep** before the exam.

- Remember your **equipment** (calculator, pencils, textbook, your notes).

- Remember that **no one exam is essential**, because your lowest hour's-worth of exam score is dropped. (This rule is intended to lower your stress, not to lower your learning! You should study for all the exams.)

- Keep matters in **perspective**. I hope you do well on the exam. I hope you enjoy and use physics throughout your life. But if you don't, this doesn't mean that you're stupid, or that you're a failure, or that you're a bad person. All of us have different talents. (I would love to learn foreign languages, and I've tried several times, but I'm just not good at learning them.) If it turns out that physics is not one of your talents, that doesn't make you evil.

4.15 Selecting a Strategy

For each problem below, find the letter of the strategy most appropriate for solving that problem.

Strategies:

A. Kinematics [e.g. $x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2$ or $v^2(t) = v_0^2 + 2a_0(x - x_0)$ or $a_{\text{circular}} = v^2/r$]
B. Force
C. Momentum conservation
D. Energy conservation
E. Energy and momentum conservation
F. Energy conservation followed by kinematics
G. Momentum conservation followed by energy conservation
H. Symmetry
I. Not solvable using what we learned in this class

Problems:

1. A ball of mass m falls through honey supplying viscous drag force $-Dv^2$. What is the terminal velocity?
2. A 130 pound woman has a stride of 2.4 feet. She stands on a still flatboat floating in still water and takes 10 steps towards shore, but she ends up only 18 feet closer to shore. How much does the flatboat weigh?
3. The Professor Street bridge over Plum Creek has a guard rail at a height of 3.2 m above the water surface. My son once placed an apple on the guard rail, then flipped it off using a hockey stick so that the apple flew off with speed 3.7 m/s at an angle of 32° above the horizontal. What was the apple's speed when it splashed into the water?
4. In one stage in stellar evolution, a rotating star shrinks and the speed of its rotation increases. This process continues until material at the star's surface rips away. During the shrinking process the product of the star's radius and its equatorial surface velocity remains constant. The surface material rips away when it experiences a threshold acceleration a_{rip}. Find a formula for the radius at which material first rips off in terms of the initial radius r_0 and the initial equatorial surface velocity v_0.
5. A ski run drops a total of 28 meters and ends with a ski jump aimed 25° above horizontal. How high does a skier fly above the ski jump launch point?
6. An empty freight car rolls at 7.1 m/s when it collides with and latches to a similar empty freight car. The latched pair then rolls down a short hill and ends up moving at 13 m/s. How high is the hill?
7. It is thought that globular galaxies result when two or more spiral galaxies collide and merge. A galaxy of 3.1 million stars traveling at 110 km/s merges with a stationary galaxy of 5.7 million stars. How fast does the resulting merged galaxy move? (All speeds are measured with respect to our own Milky Way galaxy.)

8. A lead sphere slides along a groove at 3.21 m/s. It runs into a stationary copper sphere of the same size. If the collision is elastic, what are the speeds of the two spheres after collision? (The density of lead is 11.3 g/cm^3, the density of copper is 8.9 g/cm^3.)
9. In the collision described above, both spheres emerge with dents. Which sphere has the larger dent?
10. If the copper sphere runs into the lead sphere, instead of the lead into the copper, will the sizes of the dents change?

If you found this workshop valuable, you might want to just crack open your textbook, select random problems, and think about what strategy you would use to solve them.

Chapter 5

Relativity

There were towering clouds to the south, drifting over the Hopi Reservation. The Hopis had held a rain dance Sunday, calling on the clouds — their ancestors — to restore the water blessing to the land. Perhaps the clouds had listened to their Hopi children. Perhaps not. It was not a Navajo concept, this idea of adjusting nature to human needs. The Navajo adjusted himself to remain in harmony with the universe. When nature withheld the rain, the Navajo sought the pattern of this phenomenon — as he sought the pattern of all things — to find its beauty and live in harmony with it.

— Tony Hillerman

We should take comfort in two conjoined features of nature: first, that our world is incredibly strange and therefore supremely fascinating... second, that however bizarre and arcane our world might be, nature remains comprehensible to the human mind.

— Stephen Jay Gould

5.1 The Great Race

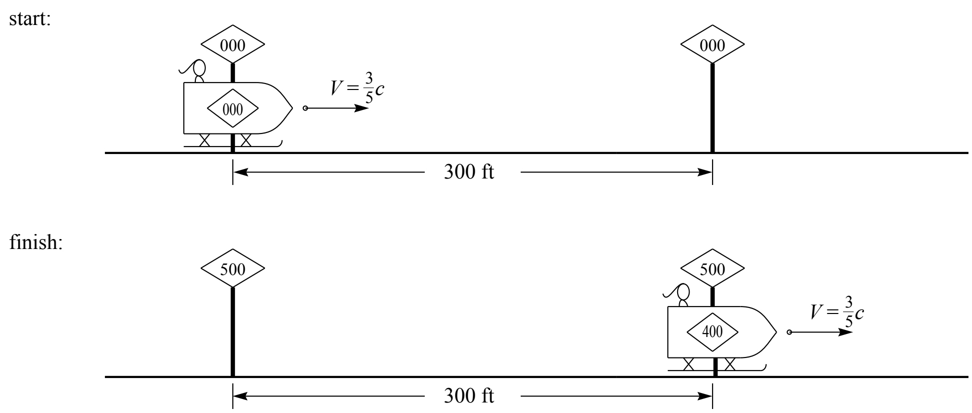
A rocket sled takes a 300 ft straightaway race at constant speed $V = \frac{3}{5}c$. (Note that $\sqrt{1-(V/c)^2} = \frac{4}{5}$.)

All clocks read time in nans. (The "nan" is the amount of time required for light to travel one foot, namely 1.017×10^{-9} seconds or about one nanosecond.)

Earth's frame

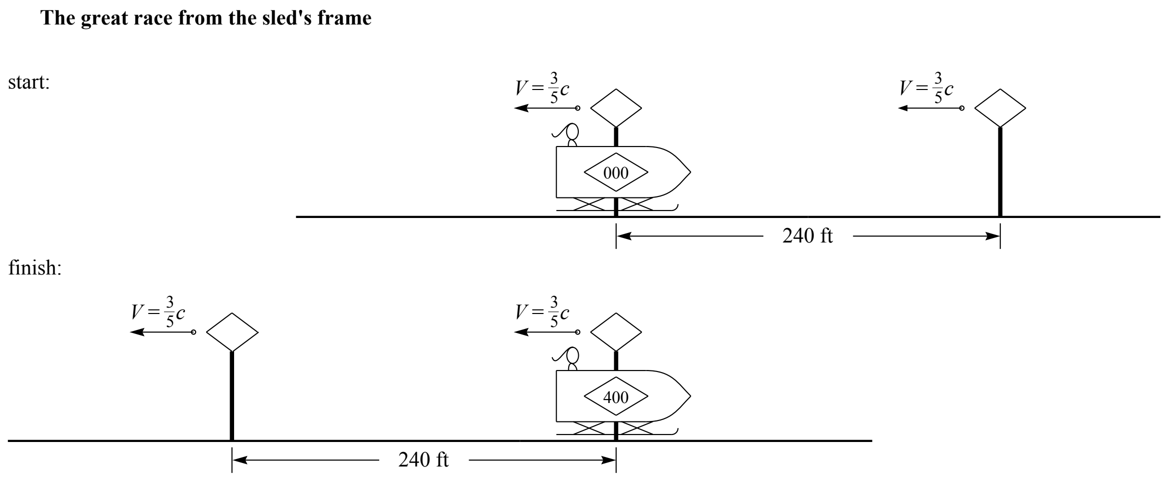
$$\text{length of racetrack} = L_0 = 300 \text{ ft}$$
$$T = \text{ time elapsed } = \frac{L_0}{V} = \frac{300 \text{ ft}}{\frac{3}{5}c} = \frac{300 \text{ ft}}{\frac{3}{5}(1 \text{ ft/nan})} = 500 \text{ nan}$$
$$T_0 = \text{ time ticked off on moving watch } = T\sqrt{1-(V/c)^2} = (500 \text{ nan})\tfrac{4}{5} = 400 \text{ nan}$$

The great race from the Earth's frame



sled's frame

$$L = L_0\sqrt{1-(V/c)^2} = (300\text{ ft})\tfrac{4}{5} = 240\text{ ft}$$



In this frame, the time elapsed is 400 nan.

The finish-line clock is moving, so it ticks off a smaller time $T\sqrt{1-(V/c)^2} = (400\text{ nan})\frac{4}{5} = 320$ nan.

¿How can the finish-line clock read 500 nan if it has ticked off only 320 nan?

5.2 Summary of Special Relativity

Time dilation	A moving clock ticks slowly.	$T = \dfrac{T_0}{\sqrt{1-(V/c)^2}}$

T_0 is the time ticked off by a single moving clock (which is also the time elapsed in that clock's own frame).
T is the (longer) time elapsed in the frame in which that clock moves at speed V.

Length contraction	A moving rod is short.	$L = L_0\sqrt{1-(V/c)^2}$

L_0 is the length of a rod in that rod's own frame (its "rest length").
L is the (shorter) length of that rod in the frame in which that rod moves at speed V.

<table>
<tr><td>Relativity of synchronization</td><td>A moving pair of clocks isn't synchronized.</td><td>Rear clock set ahead by L_0V/c^2.</td></tr>
<tr><td>Also called:
Relativity of simultaneity</td><td colspan="2">If two events are simultaneous in one frame,
then in another frame the rear event happens first.</td></tr>
</table>

If a pair of clocks is synchronized in that pair's own frame, then in the frame in which they both move at speed V, the rear (trailing) clock is set ahead by L_0V/c^2.

5.3 The Case of the Hungry Traveler

Tip: If $V = \frac{3}{5}c$, then $\sqrt{1-(V/c)^2} = \frac{4}{5}$.

Problem 1: When her wall clock reads noon, Jane Morowitz leaves her house and travels to her favorite deli at a speed of $V = \frac{3}{5}c$. Her wrist watch records 120 nans for her trip. What time does the deli's clock read when she arrives? (All three clocks involved keep excellent time.)

Solution 1: Jane's moving clock ticks slowly, so the home and deli clocks record a longer time $\frac{5}{4}(120 \text{ nan}) = 150$ nan. In the earth's frame, this is the time elapsed during the journey. (Alternatively, use the formula

$$T = \frac{T_0}{\sqrt{1-(V/c)^2}}$$

and realize that T_0 is the time ticked off on the moving single clock, Jane's clock, while T is the time obtained through the comparison of the two earth-bound clocks.)

Problem 2: In the earth's frame, how far is the deli from Jane's home?

Solution 2: It is the distance Jane traveled, namely

$$\text{distance} = \text{speed} \times \text{time} = (\tfrac{3}{5}c)(150 \text{ nan}) = \tfrac{3}{5}(1 \text{ ft/nan})(150 \text{ nan}) = 90 \text{ ft}.$$

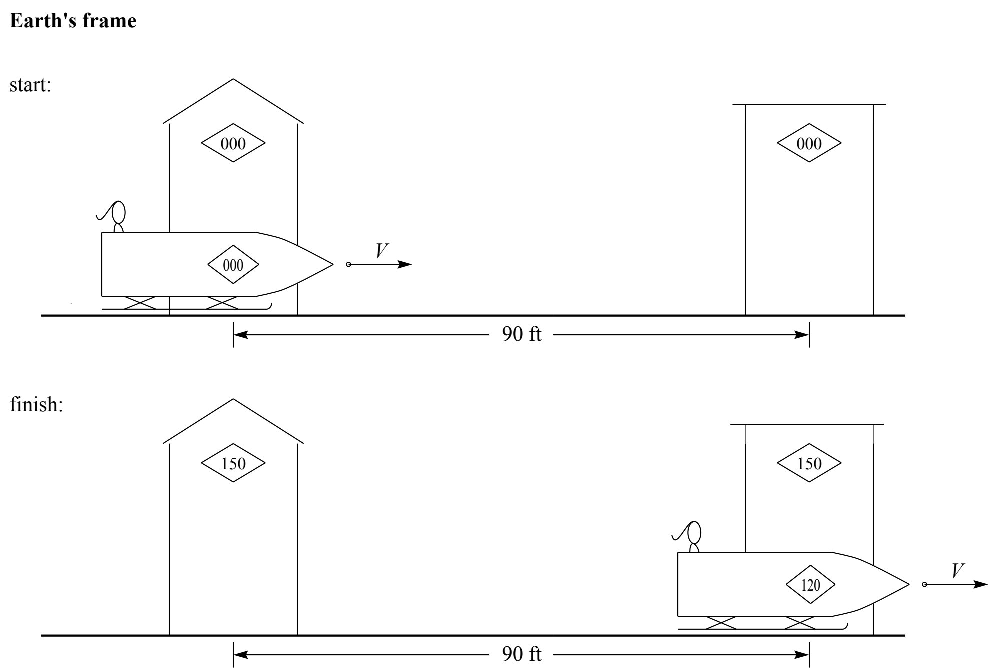


Figure 5.1: Jane's trip to the deli, in the Earth's reference frame.

The situation in the Earth's frame is summarized in figure 5.1.

Problem 3: Now we seek to understand the journey in Jane's frame. In Jane's frame, how far is the deli from Jane's home?

Solution 3(A): The distance in earth's frame (the rest frame of the two buildings) is 90 feet. The distance in Jane's frame is a length-contracted $\frac{4}{5}(90 \text{ ft}) = 72$ ft.

Solution 3(B): In Jane's frame, Jane is stationary and the deli moves towards her at speed $V = \frac{3}{5}c$. The deli requires an elapsed time of 120 nans to reach her, so it travels a distance of

$$\text{distance} \;=\; \text{speed} \;\times\; \text{time} \;= (\tfrac{3}{5}c)(120 \text{ nan}) = \tfrac{3}{5}(1 \text{ ft/nan})(120 \text{ nan}) = 72 \text{ ft}.$$

Notice that questions like "How far is the deli from Jane's home?" and "How much time did the journey take?" are ill-posed questions. One must ask instead "How much time did the journey take in the earth's frame?" or "How much time did the journey take in Jane's frame?". Because these are two distinct questions, it isn't surprising that there are two distinct answers. (Namely 150 nans and 120 nans, respectively.) ⟦The question "How far away is San Francisco?" is similarly ill-posed. One must ask instead "How far away is San Francisco from Tokyo?"⟧

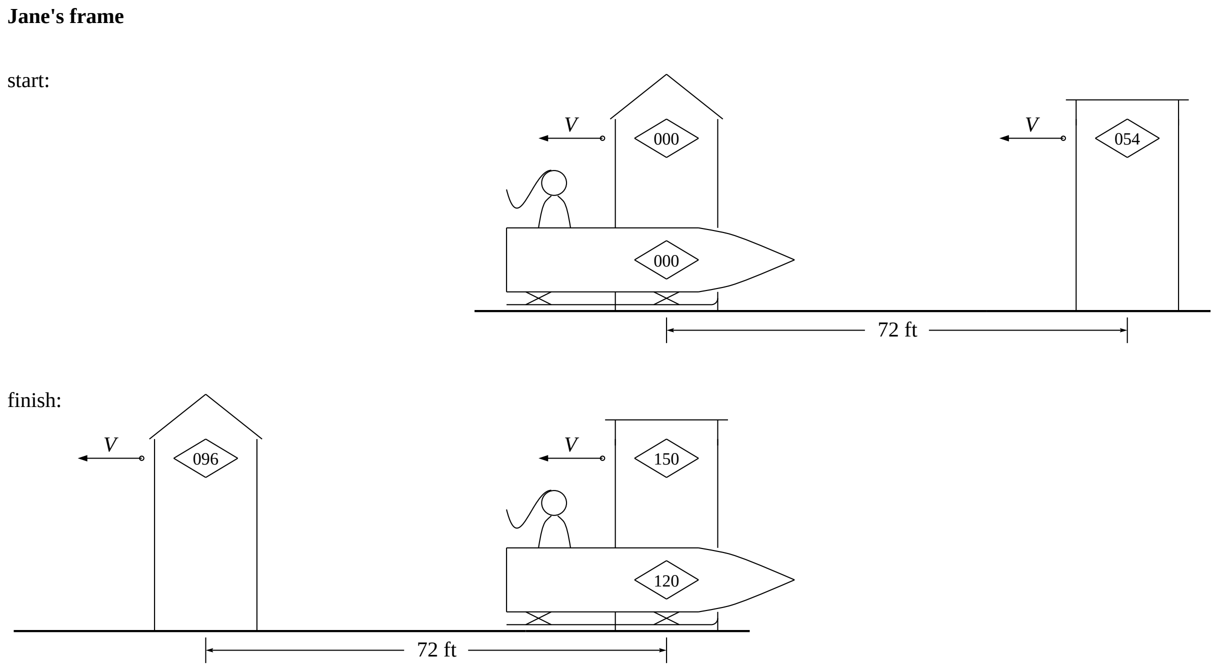


Figure 5.2: Jane's trip to the deli, in Jane's reference frame.

Problem 4: You know that Jane leaves her home at noon, travels for 120 nans (as recorded by her wrist watch), and that when she finishes her journey the deli clock reads 150 nans after noon. The deli manager finds this quite natural: "Jane, while you were traveling, your wrist watch was ticking slowly," he explains. But Jane doesn't see it that way: "Listen, bub, *I've* been stationary [in my own frame]. It's *you* who has been traveling. *Your* clock was ticking slowly all the time while it was rushing towards me. In fact, while my wrist watch ticked off 120 nans your traveling deli clock ticked off only $\frac{4}{5}(120\text{ nan}) = 96$ nan. Do you want to fight about it?" Defuse this tense situation by telling Jane the reading on the deli clock [in Jane's frame] at the instant that Jane's home wall clock struck noon.

Solution 4: In Jane's frame the deli clock (the rear clock) is set ahead of the home clock by

$$\frac{L_0 V}{c^2} = \frac{(90\text{ ft})(\frac{3}{5}c)}{c^2} = \frac{(90\text{ ft})\frac{3}{5}}{1\text{ ft/nan}} = 54\text{ nan}.$$

So, in Jane's frame, the deli clock was set to 54 nans when the journey started, then it ticked off 96 nans, so of course it reads $54\text{ nan} + 96\text{ nan} = 150\text{ nan}$ when Jane arrives.

Problem 5: In Jane's frame, what does Jane's home clock read when Jane arrives at the deli?

Solution 5: During Jane's journey her home clock ticks off 96 nans, just as the deli clock does, so it reads 96 nans after noon.

The situation in Jane's frame is summarized in figure 5.2.

> **Objection:** Hold on! Which picture is right? When Jane arrives at the deli, does her home clock read 150 nans or 96 nans?
>
> **Response:** That depends. In the earth's frame the home clock reads 150 nans when Jane arrives at the deli. In Jane's frame the home clock reads 96 nans when Jane arrives at the deli. So in the earth's frame the two events "Jane arrives at deli" and "home clock reads 150 nans" are simultaneous. But in Jane's frame those two events aren't simultaneous: the rear event ("Jane arrives at deli") happens first, and then some time later the front event ("home clock reads 150 nans") occurs. Both pictures are right.
>
> **Objection continued:** But that's contrary to common sense!
>
> **Response continued:** Of course. We don't commonly travel at $V = \frac{3}{5}c \approx 100{,}000$ miles/second.
>
> **Objection continued:** Well, maybe you're right, but suppose I ask a more concrete question. What if the deli manager looks at Jane's home clock, say with binoculars. When Jane arrives at the deli, will he see the home clock reading 150 nans or 96 nans after noon?
>
> **Response continued:** Neither. It takes some time for the light from the home clock to reach the deli manager, so the deli manager will see the "old light" that left Jane's wall clock some time ago. I'm glad you raised this objection, because it makes a great problem:

Problem 6: The deli manager watches Jane's home clock through binoculars. What time does he see when Jane arrives at the deli? Provide an analysis in both the earth's frame and Jane's frame.

Solution 6(A): (Analysis in the earth's frame.) The light travels 90 feet from Jane's home to the deli, and this requires 90 nans. Thus the manager sees the light that left the home clock 90 nans ago, when that clock read 150 nan − 90 nan = 60 nan.

Solution 6(B): (Analysis in Jane's frame.) This situation is more complicated, because the light doesn't travel 72 feet. That's because the deli manager is traveling left to meet the light emitted by the home clock, so the light travels a shorter distance. Indeed, if the time required for the light to fly from Jane's home clock to the manager's binoculars is t_f, then

$$\begin{array}{ccccc} \text{distance traveled by light} & + & \text{distance traveled by manager} & = & 72 \text{ ft} \\ ct_f & + & \frac{3}{5}ct_f & = & 72 \text{ ft} \end{array}$$

That is

$$\tfrac{8}{5}ct_f = 72 \text{ ft} \quad \text{or} \quad t_f = \tfrac{5}{8}\frac{72 \text{ ft}}{1 \text{ ft/nan}} = 45 \text{ nan}$$

Thus the light that reaches the manager at time 120 nan left Jane's home 45 nans ago. Your first impression might be that the binoculars should see a clock reading 96 nan − 45 nan = 51 nan, but this is not correct: While the light was traveling, Jane's home clock was ticking, but it didn't tick off 45 nans because this moving clock ticks slowly. Instead, it ticked off a smaller time of $(45 \text{ nan})\frac{4}{5} = 36$ nan. The binoculars see a clock reading 96 nan − 36 nan = 60 nan, which is the same result we found through our analysis in the earth's frame. The two different analyses agree on this observational consequence, as they must!

> **Objection:** What would happen if the clock emitted not light, which travels at speed c, but the newly invented "insta-rays," which (according to the inventor) travel from the home to the deli instantaneously?
>
> **Response:** In that case there would indeed be a contradiction: there would be no way to determine whether the home clock read 150 nans or 96 nans after noon. But according to special relativity, it makes no sense to say that insta-rays can send signals instantaneously: what's "instant" in one frame will require time to elapse in another frame. (In other words, two events simultaneous in one frame will not be simultaneous in another frame.)
>
> Indeed, we will see soon that no signal can travel faster than the speed of light! I advise you not to invest in the "insta-ray" corporation — according to special relativity, its product is a fraud.

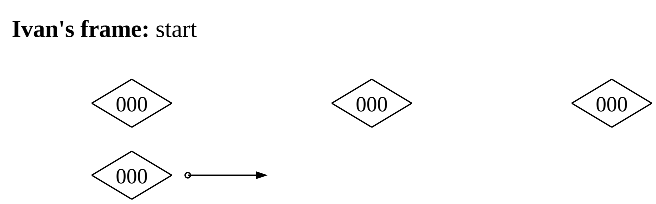
5.4 He Said, She Said

He thinks her clocks tick slowly; she thinks his clocks tick slowly: Can they ever agree?

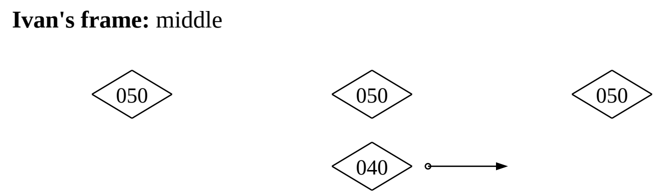
Veronica speeds past Ivan. Ivan claims that Veronica's clocks tick slowly. Veronica claims that Ivan's clocks tick slowly. Isn't this a contradiction?

It would be if time dilation were the only effect in action. But, as detailed below, because the full battery of three effects — time dilation, length contraction, and the relativity of synchronization — are all in play, there is no contradiction. (There *is*, of course, a violation of common sense.)

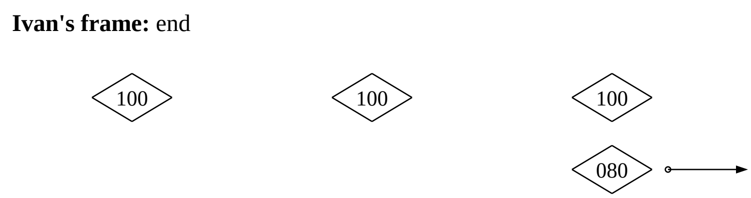
Here is the situation. (I have chosen particular speeds and times to make the numbers come out neatly. If I had used different numbers, the numerical results would be different but the same consistency would appear at the end.) Ivan has three clocks, uniformly spaced and synchronized in his frame. Veronica, carrying a watch, travels to the right past Ivan at speed $V = \frac{3}{5}c$. All the clocks read time in nans. When Veronica runs past Ivan's first clock, all four clocks read zero nans in Ivan's frame.



Fifty nans later, Veronica runs past Ivan's middle clock. Now Ivan's three clocks all read 050, but Veronica's clock has been ticking slowly by a factor of $\sqrt{1-(V/c)^2} = \frac{4}{5}$, so it reads $\frac{4}{5}(50 \text{ nan}) = 40$ nan.



In fifty more nans, Veronica runs past Ivan's last clock. Now the scene is:



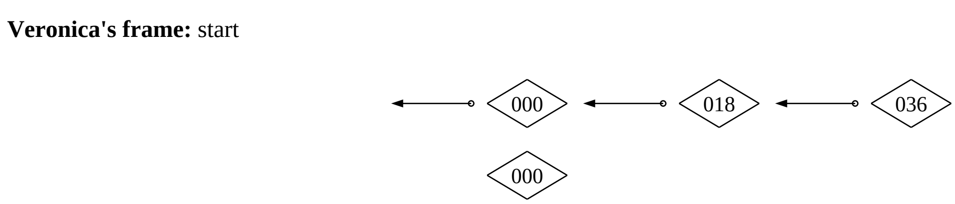
Just for fun, we can find the distance between two clocks in Ivan's frame. This is just the distance traveled by Veronica between "start" and "middle", so it is

$$\begin{aligned} \text{distance} &= \text{speed} \times \text{time} \\ &= (\tfrac{3}{5}c) \times (50 \text{ nan}) \\ &= 30 \text{ ft.} \end{aligned}$$

Now, what happens in Veronica's frame? First of all, in Veronica's frame she is stationary and Ivan's clocks are moving left at the speed of $\frac{3}{5}c$. Second, Ivan's clocks are no longer separated by the length 30 ft: because of length contraction, they are a shorter distance $\frac{4}{5}(30 \text{ ft}) = 24$ ft apart. (The "length contraction factor" is the same as the "time dilation factor," namely $\sqrt{1-(V/c)^2}$, which in this example is $\frac{4}{5}$.) Third, Ivan's three clocks are not synchronized in Veronica's frame. For example, his middle clock is to the rear of his left clock, so it is set *ahead* by a time

$$\frac{L_0 V}{c^2} = \frac{(30 \text{ ft})(\frac{3}{5}c)}{c^2} = \frac{(30 \text{ ft})(\frac{3}{5})}{1 \text{ ft/nan}} = 18 \text{ nan.}$$

Thus, the starting picture in Veronica's frame is



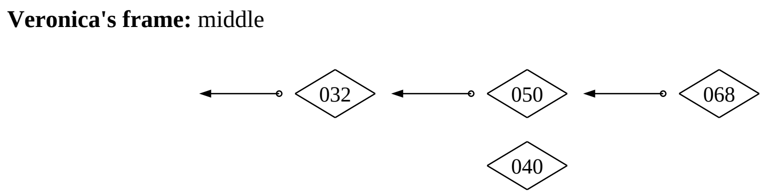
How long does it take for Ivan's middle clock to reach Veronica? (Remember that in this frame, Ivan's clock moves to reach Veronica, not vice versa.) The clock moves a (length contracted) distance 24 ft at a speed of $\frac{3}{5}c$, so the time required is

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{24\text{ ft}}{\frac{3}{5}c} = 40\text{ nan}.$$

This is the amount of time that passes, and this is the amount of time ticked off by Veronica's clock. But Ivan's clocks are moving, so they tick slowly. Each of Ivan's three clocks ticks off, not 40 nans, but the smaller time

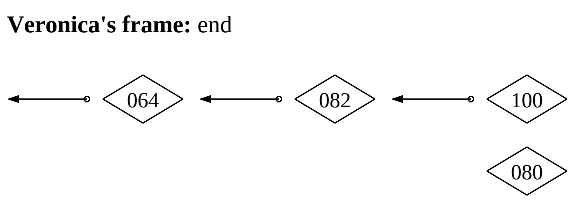
$$\tfrac{4}{5}(40\text{ nan}) = 32\text{ nan}.$$

So what is the picture in Veronica's frame at when Ivan's middle clock speeds past her? It is:



Notice the essential point: When the two clocks pass, Ivan's reads 50 nan and Veronica's reads 40 nan, *and both parties agree on this fact!*

After forty more nans pass, Ivan's last clock passes Veronica, and all three of Ivan's clocks have ticked off an additional 32 nan.



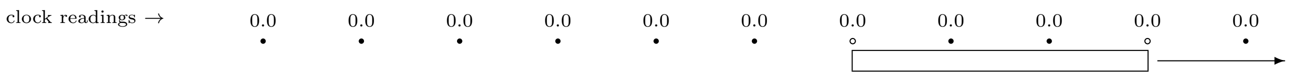
So, we return to our original question: Can Ivan and Veronica ever agree? Yes, they can agree on the essentials. They agree that whenever Veronica passes one of Ivan's clocks, the reading on Ivan's clock will be $\frac{5}{4}$ times the reading on Veronica's clock. They don't explain this in the same way: Ivan explains it by saying that Veronica's clocks tick slowly, Veronica explains it by saying that Ivan's clocks tick slowly and are out of synchronization. Although different, both explanations are correct.

This situation is similar to one from ordinary life. Residents of London and New York give different answers to the question: "How far away is Paris?" But they all agree on the answer to "How far is Tokyo from San Francisco?"

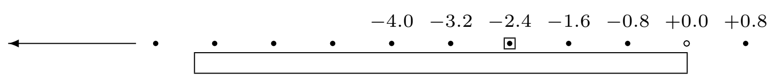
Measuring the length of a moving rod

A rod of rest length 5 feet speeds past us at $V = \frac{4}{5}c$.

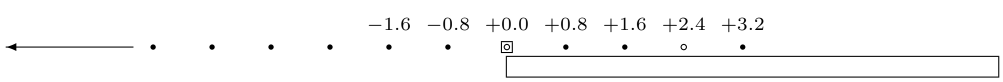
We line up observers one foot apart, synchronize their watches, and instruct them: "Raise your hand if the tip of the rod is directly in front of you at noon". The figure shows the situation in our frame when all the clocks read noon: two hands shoot up, and the distance between those hands is the length contracted amount 3 feet $= \frac{3}{5}$(5 feet).



In the rod's frame, the observers are stationed 0.6 feet apart, and adjacent clocks are out of synch by $L_0V/c^2 = 0.8$ nan. Here's the situation in the rod's frame when the right hand goes up. (Keep your eye on the third observer left of the rod's right tip. I've put a box around his dot. He is currently 1.8 feet from that tip.)



Here's the situation exactly four nans later. The lined-up observers have moved a distance of $(\frac{4}{5}c)(4 \text{ nan}) =$ 3.2 feet. Each observer's clock has ticked off $\frac{3}{5}(4 \text{ nan}) = 2.4$ nan. Thus the boxed observer's clock reads noon, and he raises his hand.



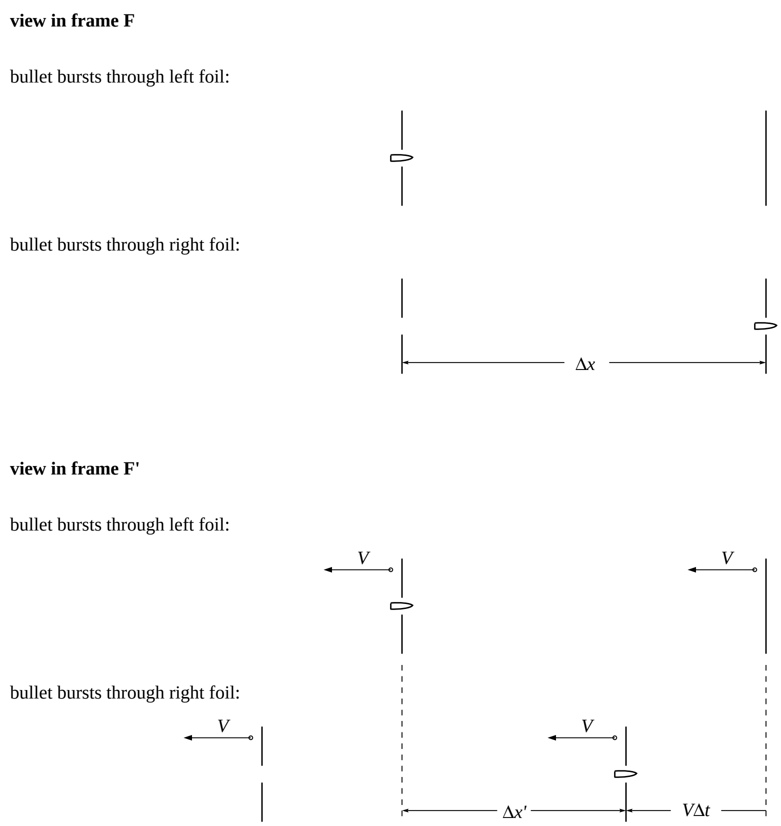
The runner says: Naturally they got a short length for my rod. They didn't raise their hands simultaneously!

Once again, the two observers explain this phenomena in different ways: The earth-bound observers say "the moving rod is short". The runner says "they didn't raise their hands simultaneously". But both observers agree that the two raised hands will have exactly two observers between them.

5.5 The Lorentz Transformation

Two events occur: for example, a bullet bursts through first one aluminum foil detector, then a second. In frame F these events are separated by distance Δx and time Δt. In frame F$'$ they are separated by distance $\Delta x'$ and time $\Delta t'$. How are these coordinates related? (The frame F$'$ is not necessarily the same as the bullet's frame. In the figures below, it is moving at a speed V that is *less* than the bullet's speed.)

Common sense:



In frame F, this process requires a time Δt. In frame F$'$, it requires an equal time $\Delta t' = \Delta t$.

The "Galilean transformation":

$$\begin{aligned} \Delta x' &= \Delta x - V\Delta t \qquad & \Delta x &= \Delta x' + V\Delta t' \\ \Delta t' &= \Delta t & \Delta t &= \Delta t' \end{aligned}$$

Correct (that is, special relativity):

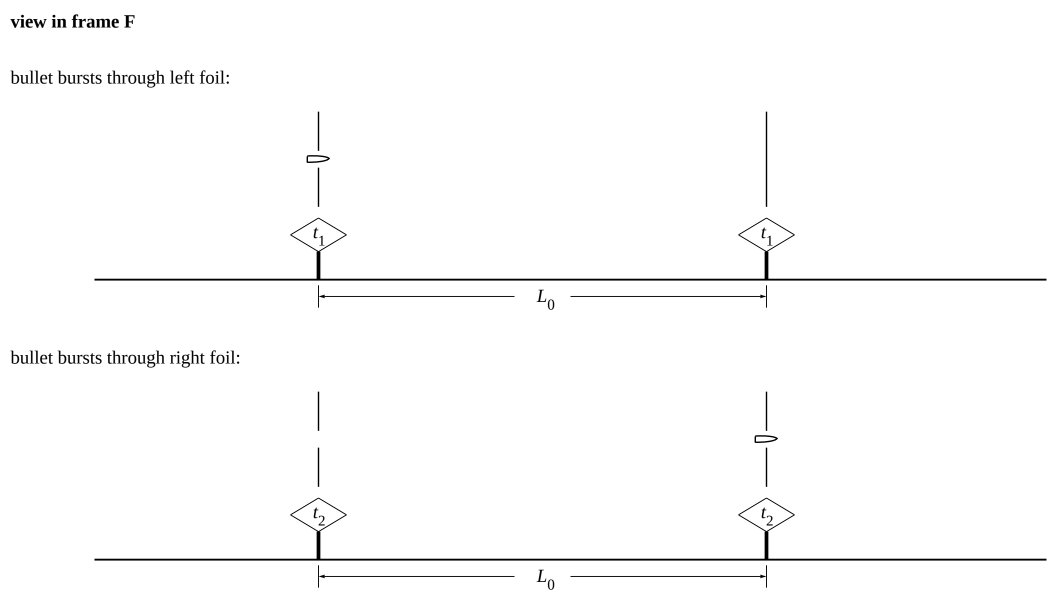
In special relativity, these relations are replaced by the "Lorentz transformation":

$$\Delta x' = \frac{\Delta x - V\Delta t}{\sqrt{1-(V/c)^2}} \qquad \Delta x = \frac{\Delta x' + V\Delta t'}{\sqrt{1-(V/c)^2}}$$

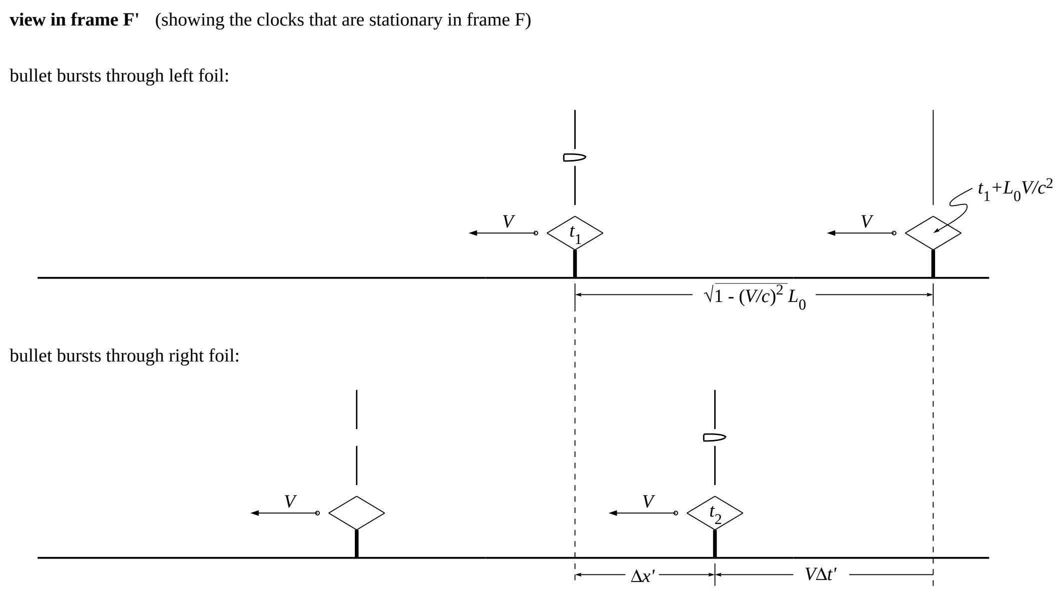
$$\Delta t' = \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}} \qquad \Delta t = \frac{\Delta t' + V\Delta x'/c^2}{\sqrt{1-(V/c)^2}}$$

Derivation of the Lorentz transformation

In frame F, the two events are separated by space $\Delta x = L_0$ and by time $\Delta t = t_2 - t_1$:



In frame F′, the two events are separated by time $\Delta t'$:



From the figure, $\Delta x' + V\Delta t' = \Delta x\sqrt{1-(V/c)^2}$ or

$$\Delta x = \frac{\Delta x' + V\Delta t'}{\sqrt{1-(V/c)^2}}.$$

How much time $\Delta t'$ has passed? The right hand clock (a single moving clock) has ticked off a time

$$T_0 = t_2 - (t_1 + \Delta x V/c^2) = \Delta t - \Delta x V/c^2.$$

But this moving clock ticks slowly: the time elapsed is not the time ticked off by the clock, it is the larger time

$$\Delta t' = \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}}.$$

5.6 The Speed of a Bullet

Suppose the two events are in fact the bullet passing through two aluminum foils. Then the speed of the bullet is v_b in frame F and v'_b in frame F′.

$$v_b = \frac{\Delta x}{\Delta t}$$

and

$$\begin{aligned} v'_b &= \frac{\Delta x'}{\Delta t'} && \text{(use Lorentz transformation to find\ldots)} \\ &= \frac{\Delta x - V\Delta t}{\Delta t - V\Delta x/c^2} && \text{(divide numerator and denominator by } \Delta t \text{ to find\ldots)} \\ &= \frac{v_b - V}{1 - v_b V/c^2} \end{aligned}$$

This is the famous "Einstein velocity addition formula."

Examples:

		common sense	Einstein formula
$v_b = 100$ mph	$V = 20$ mph	$v'_b = 80$ mph	$v'_b = 80.000\,000\,000\,000\,02$ mph
$v_b = c$	$V \neq c$	$v'_b = c - V$	$v'_b = c$
$v_b = -\frac{3}{4}c$	$V = \frac{3}{4}c$	$v'_b = -\frac{3}{2}c$	$v'_b = -\frac{24}{25}c$

5.7 Causality and Speed Limits

Given two events, it's possible that in some frames event #1 comes first, in other frames event #2 comes first, and in one frame the two events are simultaneous.

But suppose that event #1 causes event #2. (For example: event #1 is "I toss a snowball," event #2 is "snow splatters over the wall.") In this case you'd certainly think that event #1 has to occur before event #2 in all frames! Let's make this assumption and see where it takes us.

Define

$$\frac{\Delta x}{\Delta t} = \text{ speed of the causal signal } = v_s.$$

(In our example, the "causal signal" is the snowball.)

Our assumption is $\Delta t' > 0$ in all reference frames, so, by the Lorentz transformation,

$$\Delta t' = \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}} > 0 \quad \text{for all frames.}$$

Thus

$$\begin{aligned} \Delta t &> V\Delta x/c^2 \\ 1 &> V v_s/c^2 \\ c^2 &> V v_s \end{aligned}$$

This holds for all frames, and for all causal signals.

Suppose the causal signal is light, so that $v_s = c$. Then

$$c > V,$$

that is, all reference frames travel at less than the speed of light.

Pick one such reference frame moving just slower than c: that is $V = c - \epsilon$, where ϵ can be as small as you wish as long as it's greater than zero. Then

$$\begin{aligned} c^2 &> V v_s \\ c^2 &> (c - \epsilon) v_s \\ c &> \left(1 - \frac{\epsilon}{c}\right) v_s \end{aligned}$$

Let $\epsilon \to 0$ to find

$$c \geq v_s,$$

that is, any causal signal travels at less than or equal to the speed of light.

5.8 Rigidity, Straightness, and Strength

Here is a proposed technique for sending a signal faster than the speed of light... in fact, for sending it instantaneously: Push the left end of a rod... the right end moves at the same time! Well, not quite. When you push the left end of the rod, you move the first atom in a long chain of atoms that makes up the rod. A short time later, the first atom pushes the second, then the second pushes the third, and so forth. This push moves down the rod and reaches the end at a speed that is very fast by human standards[1], so we don't notice it. But the speed is very slow compared to the speed of light. *There is no such thing as a perfectly rigid rod.*

Ivan mounts a straight rod horizontally on three pegs, and he fits the base of each peg with a firecracker that can cause it to crumble into bits. He arranges for the three pegs to crumble simultaneously, and at that same instant the rod begins to fall down. The rod is always straight and always horizontal. In Veronica's reference frame these three events are not simultaneous: First the right peg crumbles and the right end of the rod begins to fall, second the middle peg crumbles and the middle of the rod begins to fall, third the left peg crumbles and the left end of the rod begins to fall. Between the first and second events the rod must be curved in Veronica's reference frame. *A rod that is straight in one reference frame may be curved in another.* (This prediction has been tested experimentally, not with regular rods, but with electric field lines: Field lines that are straight in one reference frame may be curved in another.)

A bicycle wheel is set up on a rack and spun very quickly. The pieces of the wheel are moving in the direction that they are pointing, so they are length contracted. But the spokes are moving in a direction

[1] For a steel rod it moves at about 3 miles/second.

perpendicular to the direction that they are pointing, so they are not length contracted. How can wheel hold together with a contracted circumference and a non-contracted radius? The answer is that it can't. When a wheel rotates fast it breaks apart and flies into pieces. A wheel made of a weak material like wood will break apart at rather slow speeds; a wheel made of a strong material like steel will break apart at higher speeds; a wheel made of a very strong material like diamond will break up at still higher speeds; but a wheel made up of *any* material will break apart a speeds much lower than those where the relativistic effect becomes noticeable. *There is no infinitely strong material.*

The special relativistic limits on the rigidity and strength of materials are can be worked out quantitatively,[2] and they are extreme. All known materials are much less rigid and much less strong than the limits allow.

5.9 Puzzles and Paradoxes

There are no paradoxes in relativity. The set of Lorentz transformations constitutes a mathematical group, and the (straightforward) proof of this assertion proves that there are no contradictions within relativity.

But there are contradictions between relativity and common sense, and when you're working on a specific problem it's easy to let common sense creep in, creating an apparent paradox. Unmasking such contradictions by ferreting out exactly where the common sense crept into your assumptions is a great way to learn relativity and to develop your relativistic intuition.

Warm up questions

1. Veronica speeds past Ivan. He says her clocks tick slowly, she says his clocks tick slowly. This is not a logical contradiction because

a. Ivan observes the hands of Veronica's clocks as length contracted.

b. Veronica compares her clock to two of Ivan's clocks, and those two clocks aren't synchronized.

c. two events simultaneous in Ivan's frame are always simultaneous in Veronica's frame as well.

d. a moving rod is short.

Note: There is nothing logically inconsistent about both clocks ticking slowly. You know that a person standing in Los Angeles thinks (correctly!) that Tokyo is below his feet, while a person standing in Tokyo thinks (correctly!) that Los Angeles is below his feet. This is not a logical contradiction and you are familiar with it. It is just as true that Ivan thinks (correctly!) that Veronica's clock ticks slowly, while Veronica thinks

[2] Wolfgang Rindler, *Introduction to Special Relativity* (Clarendon Press, Oxford, 1982).

(correctly!) that Ivan's clock ticks slowly. This is not a logical contradiction but you are not familiar with it. Through this course, you are becoming familiar with "this strange and beautiful Universe, our home."[3]

2. Ivan says Veronica's rods are short, Veronica says Ivan's rods are short. This is not a logical contradiction because

a. a moving clock ticks slowly.

b. Ivan's clocks tick slowly, so by distance = speed × time, the distance must be smaller too.

c. it takes some time for light to travel the length of the meter stick.

d. two events simultaneous in Ivan's frame may not be simultaneous in Veronica's.

3. A moving clock ticks slowly because

a. time passes slowly in the moving frame.

b. the clock was damaged during acceleration.

c. the observer is looking at "old light" which required a finite time to get from the clock to the observer.

4. A pair of clocks is initially synchronized. Each clock undergoes an identical acceleration program until both clocks are moving at constant speed $0.9c$. The two clocks fall out of synchronization because

a. the rear clock has been moving for longer, so its reading falls behind that of the front clock.

b. the front clock has been moving for longer, so its reading falls behind that of the rear clock.

c. during the acceleration process, the phenomena of general relativity are in play. (We will study this in the last class of the semester.)

Paradoxes of space and time

1. *Boxcar with a bomb.*[4]
A railroad boxcar has a clock mounted on its front wall and a clock mounted on its rear wall. Wires run from these clocks to a package at the middle of the boxcar, and each clock sends an electrical signal to the package at the instant that the clock strikes noon. Within the package is a bomb set to explode if it gets a signal from one clock or from the other, but *not* if it gets a signal from both. In the boxcar's frame the two clocks are synchronized, so an analysis in the boxcar's frame indicates that the bomb will not explode. But in the earth's frame the two clocks are not synchronized, so an analysis in the earth's frame indicates

[3] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, 1973), page v.
[4] This paradox was inspired by a question from Roth Reason, class of 2000.

that the bomb *will* explode. Clearly, these two analyses cannot both be correct. Find the flaw (a hidden assumption) in one analysis, and state definitively whether the bomb explodes or not.

2. *Train in a tunnel.*[5]
A train with skylights has the same rest length as a tunnel. The train goes through the tunnel at high speed. In the train's frame, the tunnel is contracted. . . as the train passes through two passengers, one at the front and one at the back, glance up at the same moment, and both of them see blue sky. In the tunnel's frame, the train is contracted. How can the passengers look up and both see blue sky? *Clue:* The resolution requires two sketches in the tunnel's frame, one in the train's frame.

3. *Busted bus.*
A bus 30 feet long in its own reference frame moves so fast that it is only 6 feet long in the earth's reference frame. It speeds through a "falling rock zone" and, as bad luck would have it, a rock (15 feet long in its own frame) rolls down the mountainside and completely crushes the 6-foot-long bus.

Or does it? In the reference frame of the bus, the rock is 3 feet long. Does the rock punch a neat 3-foot hole in the roof of the 30-foot bus?

This is not a matter about which we can say "both are right": the bus is either completely squashed or else it is damaged but not destroyed. Which analysis is correct?

4. *Pole in the barn.*
Most barns have two doors, so that you can drive a trailer into the barn, stop and unload it, and then drive it out without backing up. I grew up on a farm in Bucks County, Pennsylvania, where we had a barn with exactly 100 feet of width separating its two doors.

One day a world champion pole vaulter came to visit our farm. He carried his favorite pole which, by coincidence, was also exactly 100 feet long. The champion boasted that he was so fast that, even carrying his pole horizontally, he could run right through our barn at the speed of $V = \frac{4}{5}c$.

"At that speed," he assured my father, "my pole will be length contracted until it's only $\frac{3}{5}$(100 feet) = 60 feet long. I'll be able to fit it completely within your old barn! Look, if you don't believe it, put me to the test. Station one of your sons at the front door and the other one at the back door. Start with both doors closed, and open each one for just long enough to allow me through. You'll see. There will be a time when both doors are shut and my pole is completely enclosed within your barn."

My father was no dummy. He rubbed his chin and looked puzzled and thoughtful for a minute. "Okay, you're on," he told the pole vaulter. "I'll station my boys. But there's just one thing I don't understand: Sure, in the barn's frame your pole will be length contracted. But in *your* frame the *barn* is moving. In *your* frame my barn is 60 feet wide and your pole is 100 feet long. How are you going to fit that long pole of yours into my stubby little barn?"

Now it was the champion's turn to be puzzled. In fact, he looked frightened and just a little greenish. I could see the sweat bead up on his forehead, and he lost his confident swagger. He wanted to bail out. My

[5]Due to Wendy Jackson, class of '03, and Marta Johnson, class of '04.

older brother went up and whispered a few words into the vaulter's ear. They huddled in quiet conversation for a few minutes, and then the vaulter regained most of his lost confidence. He carried out the feat flawlessly.

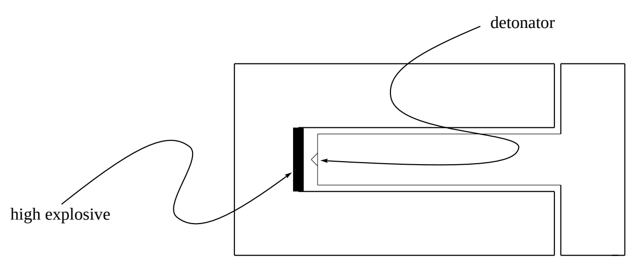
What did my brother tell the vaulter?

5. *Falling hula hoop.*
A horizontal hula hoop, one yard wide in its own frame, falls slowly from the ceiling. As it falls, a horizontal yard stick is launched across the room so fast that it is only one foot long in the room's frame. The launch is timed so that the hula hoop momentarily surrounds the shrunken yard stick. The length-contracted yard stick easily passes through the center of the hula hoop.

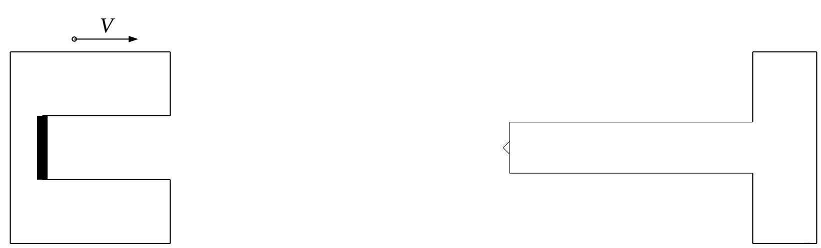
Consider this from the yard stick's frame. Now the hula hoop is length-contracted. How can a three-foot long yard stick pass through the center of a one-foot wide hula hoop?

Paradoxes concerning objects traveling at high speeds

6. *Explosion!*[6]
I have a U-shaped piece of hardened steel, and I place high explosive into the hole. I also have a T-shaped piece of the same hardened steel, and on the tip of the T I mount a detonator. The T is just short enough that I can gently slide the T into the U and the detonator will not touch the explosive.

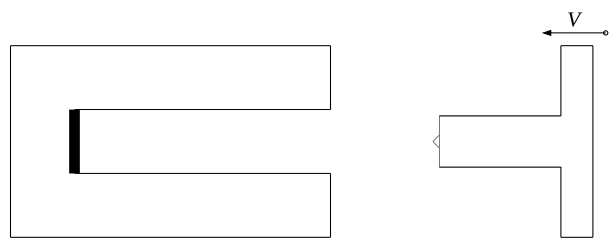


Now I hurl the U towards the T at a substantial speed. The U is length contracted, so when the two pieces come together the detonator and explosive touch. The U and T are both destroyed in the resulting explosion.



[6]Source: E.F. Taylor and J.A. Wheeler, *Spacetime Physics*, second edition (W.H. Freeman, New York, 1992) page 185.

But wait! Examine the same process from the reference frame of the U. In this frame the T is length contracted, the detonator does not touch the explosive, and both pieces remain intact.



Clearly, these two analyses cannot both be correct. Find the flaw (a hidden assumption) in one analysis, and state definitively whether the explosion goes off or not.

7. *Stealth tank.* A tank is an important military vehicle for two reasons: First, it carries a cannon. Second, it can travel over rough terrain dotted with holes and ditches.

A military contractor approaches a pentagon procurement officer with a proposal to build a "stealth tank" that can travel at relativistic speeds. The contractor points out all the advantages of the tank: its armament, its speed, its evasiveness. Then he comes to the clinching point of his sales pitch: "Officer, this tank is 10 meters long in its own frame, so if it were an ordinary tank attempting to go over a 10 meter wide hole, it would fall in. But it moves so fast that in its reference frame, that hole is length contracted to be only 2 meters wide. Our tank will roll right over that little hole!"

The procurement office has heard similar extravagant claims from other contractors, so she thinks about it for a while. "Wait a minute," she says, "in the reference frame of the hole, the hole is 10 meters wide and the tank is length contracted to be only 2 meters long. I'm sure that your stubby little tank will be trapped in such a wide hole."

Who's right, the contractor or the procurement officer?

8. *Joust.*[7]
Sir Lancelot and Sir Mordred each has a lance of rest length 5 meters. During a joust, each gallops towards the other at a speed of $\frac{3}{5}c$ relative to the earth. (For definiteness, assume that Lancelot gallops to the right, $v_L = +\frac{3}{5}c$, while Mordred gallops to the left, $v_M = -\frac{3}{5}c$.)

a. The crowd in the stands sees a fair fight, because each knight has a lance of equal length. How long are these two lances in the Earth's frame?

b. How fast is Mordred moving in Lancelot's frame? How long is Mordred's lance in Lancelot's frame?

c. How fast is Lancelot moving in Mordred's frame? How long is Lancelot's lance in Mordred's frame?

[7]Based on the documentation for computer program *RelLab*: Paul Horwitz, Edwin Taylor, and Kerry Shetline (Physics Academic Software) pages 68–69.

In Lancelot's frame, his own lance is much longer than his opponent's, so Lancelot expects to win easily. However, the same holds for Mordred! Meanwhile, in the crowd's frame, the two lances are equally long.

In the crowd's frame, each lance makes contact with the opposing knight's breastplate simultaneously. Also, both knights are unhorsed simultaneously. (For definiteness, call "Mordred unhorsed" event #1 and "Lancelot unhorsed" event #2.)

d. Which knight is unhorsed first in Lancelot's frame? In Mordred's?

e. Is Lancelot unhorsed at the instant that his breastplate makes contact with Mordred's lance?

f. In each knight's frame, one knight is unhorsed by another knight who is already on the ground. How can a knight on the ground succeed in unhorsing his opponent?

5.10 Dynamics in Special Relativity

We've been talking about things like space and time. Will this have any effect on things like force, momentum, and energy? Of course!

1. How does force affect motion?

Newton: A body subject to constant force F will have velocity $v = (F/m)t$, which increases without bound when t increases.

Einstein: But v can't exceed c! Newton's formula, although an excellent approximation for small velocities, must be wrong.

2. What is the origin of force?

Newton: The gravitational force on the Earth due to the Sun is

$$G\frac{m_E m_S}{r^2}.$$

Einstein: This formula says that if you move the Sun, the gravitational force on the Earth changes instantly! Relativity demands a time delay of about eight minutes. Newton's formula must be wrong.

We will begin with the "How does force affect motion?" question. The "What is the origin of force?" question is the subject of most of modern field theory, including Einstein's theory of general relativity, which we'll treat very briefly on the last day of class. [But let me recommend the book "Einstein's unfinished symphony: listening to the sounds of space-time" by Marcia Bartusiak (Joseph Henry Press, Washington, DC, 2000).]

Derivations

Remember that the word "derivation" is a misnomer in physics. Instead we're trying to make sensible assumptions and definitions, consistent with the Lorentz transformation, so that the results of our assumptions and definitions can be tested in the laboratory. But of course it is nature, and not we, who decides what is "sensible".

Indeed, Einstein made a clunky definition of force in his 1905 paper that gave birth to relativity, and it's not the one used today. [See *The Principle of Relativity* (Dover, 1952), note by Arnold Sommerfeld on page 63.]

In this course we'll deal with the one-dimensional situation: a particle moves in the x-direction, subject to forces in the x-direction.[8]

In the earth's frame, at some particular instant, the particle moves at velocity v. Because the particle is subject to a force, a moment later it will be moving at a different velocity. But that's okay. We'll consider the inertial frame that moves at velocity $V = v$, that is, the inertial frame in which the particle is (temporarily) at rest. This is called the particle's "temporary rest frame". Now in the temporary rest frame the particle is stationary, and even a moment later when the particle's not stationary it will be moving slow enough that it's legitimate to apply classical mechanics in that frame. We'll find the consequences of $F_{\text{net}}^{\text{trf}} = m^{\text{trf}} a^{\text{trf}}$ in the temporary rest frame, where this formula is legitimate, and then transform back to the earth's frame. (Which is, after all, where our measuring apparatus is located.)

In order to carry out this strategy, we need the frame transformation equations for force, mass, and acceleration. The equation for acceleration can be derived from our study of space and time in relativity. The derivation (see appendix A) shows that

$$a^{\text{trf}} = \frac{1}{(\sqrt{1-(v/c)^2})^3} a^{\text{earth}}.$$

On the other hand, there's no way to *derive* the transformation equations for mass and for force. We just have to make an assumption (one that's consistent with the Lorentz transformation), find the consequences of that assumption, and see whether that assumption gives rise to results in accord with experiment. The assumption we'll make is:

> The mass of a particle in any reference frame is defined as the mass in its temporary rest frame.
> The force acting on a particle in any reference frame is defined as the force acting in its temporary rest frame.

(It turns out that the assumption concerning mass is effective for three-dimensional as well as one-dimensional motion, but the assumption concerning force is effective only for one dimension.)

[8] For forces not parallel to the x-axis, see Robert Resnick, *Introduction to Special Relativity*, page 125, or John B. Kogut, *Introduction to Relativity*, pages 67–73. However, I recommend that you put off this reading until after you take a course on Electricity and Magnetism.

Given these assumptions, the equation of motion in the earth's frame (or any inertial frame, for that matter) is

$$F_{\rm net} = \frac{m}{(\sqrt{1-(v/c)^2})^3} a.$$

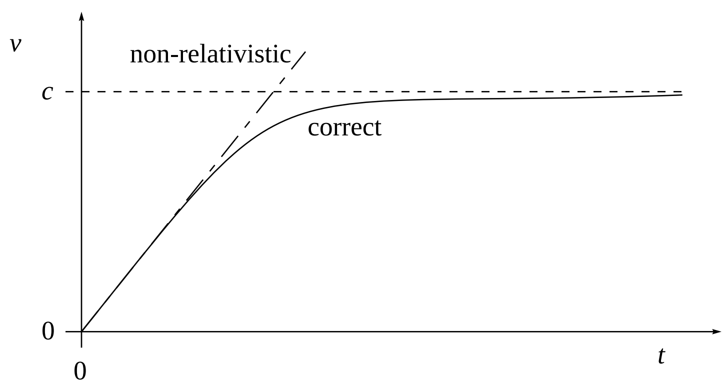
No longer is the inertia equal to the mass. Instead, the inertia is equal to $m/(\sqrt{1-(v/c)^2})^3$, which increases with velocity until it approaches infinity as v approaches c.

Can we find consequences of this equation, consequences that can be compared to experiment? Indeed we can. What about the simplest possible case, motion subject to constant force? In classical mechanics, the velocity increases without limit,

$$v = Ft/m.$$

In relativistic mechanics, the velocity is given (see appendix B) by

$$v = \frac{Ft/m}{\sqrt{1+(Ft/mc)^2}}.$$



It is difficult to test this prediction for baseballs falling subject to gravity, but straightforward to test it for electrons moving subject to a uniform electrical force. The relativistic result is borne out. At Fermilab, protons are accelerated with such force for such a long time that they would be moving at $v = 50c$ if common sense were correct. But instead they move always with v just a tad less than c, exactly as the graph above dictates.

Well, that's force. What about momentum and energy? [When I talk about energy here, I'm talking about energy involved with motion — kinetic energy, we called it back in classical mechanics. The concept of potential energy — or field energy as it's called in relativity — is closely connected with the "What is the origin of force?" question that we're not treating in this course. Put in another way, the E here is the energy of the particle only: if the particle interacts (e.g. through gravity or electromagnetism) then there is also potential energy, but that potential energy is associated with the field (e.g. gravitational field or electromagnetic field), not with the particle.] These issues are discussed in appendices C and D. The results are

$$p = \frac{mv}{\sqrt{1-(v/c)^2}}, \qquad E = \frac{mc^2}{\sqrt{1-(v/c)^2}}.$$

It's easy to see how to interpret the result for momentum — relativistic momentum is similar to classical momentum mv, with a difference that is negligible when $v \ll c$. But how are we to interpret this strange formula for E, which seems to have no relation to the familiar classical kinetic energy $\frac{1}{2}mv^2$?

What is the non-relativistic (i.e. $v \ll c$) limit of the expression for E? Use the fact that for small values of ϵ (i.e. $|\epsilon| \ll 1$)

$$(1+\epsilon)^n \approx 1 + n\epsilon.$$

(This constitutes the first two terms of the Taylor series expansion for $f(x) = (1+x)^n$ about $x = 0$.) Apply this formula to the expression for E, using $\epsilon = -(v/c)^2$ and $n = -\frac{1}{2}$. This gives

$$E \approx mc^2[1 + \tfrac{1}{2}(v/c)^2] = mc^2 + \tfrac{1}{2}mv^2.$$

In this limit, the relativistic energy is just the familiar classical kinetic energy, $\frac{1}{2}mv^2$, plus a constant "shift in zero-of-energy" term, the so-called "rest energy" mc^2.

Does this rest energy term have any physical manifestation? Yes it does. Suppose two wads of bubble gum, each of mass m, approach each other, each at speed $\frac{3}{5}c$ with respect to the earth. They collide and stick to each other in a stationary wad of mass M. Momentum is conserved: the total momentum is zero before the collision and zero after. But energy is also conserved:

$$\frac{mc^2}{\frac{4}{5}} + \frac{mc^2}{\frac{4}{5}} = Mc^2$$

so

$$\tfrac{5}{2}m = M.$$

What? What's that you say? You say that if you put together two wads, each of mass m, you get a wad of mass $2.5\,m$? Yes, that's exactly what I say. In relativity, the mass of a composite is not equal to the sum of the masses of the constituents. Instead, the energy of the composite is equal to the sum of the energies of the constituents. This has been tested experimentally numerous times — see appendix E. In the vernacular, we call this "converting energy into mass" through the equation for rest energy,

$$E = mc^2,$$

an equation you might have encountered previously.

Summary of results

A bullet of mass m moves parallel to the x-axis. Forces parallel to the x-axis are applied. As seen from a single reference frame F, we have:

If the sum of forces acting on the bullet is F_b, then

$$F_b = \frac{ma_b}{(\sqrt{1-(v_b/c)^2}\,)^3}.$$

Thus the bullet's inertia is no longer the same as its mass: Its inertia is

$$\frac{m}{(\sqrt{1-(v_b/c)^2}\,)^3},$$

which grows to infinity as its velocity approaches c.

The relativistic momentum of the bullet is

$$p_b = \frac{mv_b}{\sqrt{1-(v_b/c)^2}}.$$

and its relativistic energy (energy involved with motion) is

$$E_b = \frac{mc^2}{\sqrt{1-(v_b/c)^2}}.$$

In classical collisions, momentum is conserved, the sum of the particle masses is conserved, and kinetic energy might or might not be conserved. In relativistic collisions, momentum is conserved, energy is conserved, and the sum of the particle masses might or might not be conserved.

What are these quantities as observed in frame F′, which moves along the x-axis at speed V? They are

$$\begin{aligned} F'_b &= F_b \\ p'_b &= \frac{p_b - VE_b/c^2}{\sqrt{1-(V/c)^2}} \\ E'_b &= \frac{E_b - Vp_b}{\sqrt{1-(V/c)^2}}. \end{aligned}$$

The combination

$$E_b^2 - (p_b c)^2,$$

the "invariant," is the same in all reference frames. Furthermore, if I make E the total energy (the sum of the energies of each relevant particle) and p the total momentum (the sum of the momenta of each relevant particle), then

$$E^2 - (pc)^2$$

is the "conserved invariant." It is the same for all times and in all reference frames.

Appendix A: Transformation of accelerations in special relativity

$$\text{In frame F:} \quad a_b = \frac{dv_b}{dt}.$$

$$\text{In frame F}': \quad a_b' = \frac{dv_b'}{dt'}.$$

How are these related? By the chain rule

$$\frac{dv_b'}{dt'} = \frac{dv_b'}{dt}\frac{dt}{dt'}.$$

But

$$v_b' = \frac{v_b - V}{1 - v_b V/c^2}$$

so

$$\begin{aligned} \frac{dv_b'}{dt} &= \frac{(1 - v_b V/c^2)\frac{dv_b}{dt} - (v_b - V)\left(-\frac{dv_b}{dt}\frac{V}{c^2}\right)}{(1 - v_b V/c^2)^2} \\ &= \frac{(1 - v_b V/c^2) + (v_b V/c^2 - V^2/c^2)}{(1 - v_b V/c^2)^2}\left(\frac{dv_b}{dt}\right) \\ &= \frac{1 - V^2/c^2}{(1 - v_b V/c^2)^2}\left(\frac{dv_b}{dt}\right). \end{aligned}$$

Meanwhile

$$t' = \frac{t - V x_b/c^2}{\sqrt{1 - (V/c)^2}}$$

so

$$\frac{dt'}{dt} = \frac{1 - V(dx_b/dt)/c^2}{\sqrt{1 - (V/c)^2}} = \frac{1 - v_b V/c^2}{\sqrt{1 - (V/c)^2}}$$

whence

$$\frac{dt}{dt'} = \frac{\sqrt{1 - (V/c)^2}}{1 - v_b V/c^2}.$$

Together

$$\frac{dv_b'}{dt'} = \frac{dv_b'}{dt}\frac{dt}{dt'} = \frac{(1 - (V/c)^2)^{3/2}}{(1 - v_b V/c^2)^3}\left(\frac{dv_b}{dt}\right)$$

or

$$a_b' = \left(\frac{\sqrt{1 - (V/c)^2}}{1 - v_b V/c^2}\right)^3 a_b.$$

This expression is pretty formidable. *But* if the frame F$'$ is (temporarily) moving along with the bullet, i.e. $V = v_b$, then

$$a_b' = \frac{1}{(\sqrt{1 - (v_b/c)^2})^3} a_b.$$

Appendix B: Motion with a single constant force

In this case in nonrelativistic mechanics, $v = (F/m)t$. What about in relativity?

$$\begin{aligned}
F &= \frac{m\,dv/dt}{[1-(v/c)^2]^{3/2}} \\
\frac{F}{m}dt &= \frac{dv}{[1-(v/c)^2]^{3/2}} \qquad [\![\beta = v/c]\!] \\
\frac{F}{mc}dt &= \frac{d\beta}{[1-\beta^2]^{3/2}} \\
\frac{F}{mc}t &= \int_0^{v/c} \frac{d\beta}{[1-\beta^2]^{3/2}} = \left[\frac{\beta}{[1-\beta^2]^{1/2}}\right]_0^{v/c} = \frac{v/c}{\sqrt{1-(v/c)^2}}.
\end{aligned}$$

Now it is a simple matter to solve for v, finding

$$v = \frac{Ft/m}{\sqrt{1+(Ft/mc)^2}}.$$

Appendix C: Relativistic momentum

Define through

$$\frac{dp}{dt} = F_{\text{net}}.$$

But

$$F_{\text{net}} = \frac{m}{[1-(v/c)^2]^{3/2}}a$$

and

$$\frac{dp}{dt} = \frac{dp}{dv}\frac{dv}{dt} = \frac{dp}{dv}a,$$

so

$$\frac{dp}{dv} = \frac{m}{[1-(v/c)^2]^{3/2}}.$$

Integrate both sides from 0 to v.

$$\begin{aligned}
p(v) - p(0) &= \int_0^v \frac{m}{[1-(v/c)^2]^{3/2}}\,dv \\
&= mc\int_0^{v/c} \frac{1}{[1-\beta^2]^{3/2}}\,d\beta \\
&= mc\left[\frac{\beta}{[1-\beta^2]^{1/2}}\right]_0^{v/c} \\
&= \frac{mv}{\sqrt{1-(v/c)^2}}.
\end{aligned}$$

Thus

$$p = \frac{mv}{\sqrt{1-(v/c)^2}}.$$

Appendix D: Relativistic energy

Define relativistic energy involved with motion through

$$\frac{dE}{dx} = F_{\text{net}}.$$

But

$$F_{\text{net}} = \frac{m}{[1-(v/c)^2]^{3/2}}a$$

and

$$\frac{dE}{dx} = \frac{dE}{dv}\frac{dv}{dt}\frac{dt}{dx} = \frac{dE}{dv}\frac{a}{v},$$

so

$$\frac{dE}{dv} = \frac{mv}{[1-(v/c)^2]^{3/2}}.$$

Integrating both gives

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}}.$$

Appendix E: Mass in relativity

Is the mass of a composite object equal to the sum of the masses of its constituents? The answer "yes" seems so natural and obvious that the question hardly needs asking. Yet relativity claims that the correct answer is "no"! (Instead, the energy of the composite is equal to the sum of the energies of its constituents.) As always, the test of correctness is experiment, not obviousness.

The masses of atoms and subatomic particles have been measured to very high accuracy (primarily through the technique of "mass spectroscopy"). For example, the mass of the proton is known to 11 significant digits. In this document, I'll present only a handful of the many measurements available, and I'll round them down to seven decimal places, which is more than enough accuracy to prove my point. The masses here are given not in terms of the kilogram (abbreviated as "kg") but in terms of the "atomic mass unit" (abbreviated as "u"), which is 1/12 the mass of a carbon-12 atom ($^{12}_{6}$C). (These data come from the National Institute of Standards and Technology through < http://physics.nist.gov/cuu/Constants/index.html > and from the Atomic Mass Data Center in Orsay through < http://www-nds.iaea.org/amdc/ >.) These sources give the mass values:

mass of electron	0.000 548 5 u
mass of proton	1.007 276 4 u
mass of $^{1}_{1}$H	1.007 825 0 u
mass of neutron	1.008 664 9 u
mass of $^{4}_{2}$He	4.002 603 2 u
mass of $^{8}_{4}$Be	8.005 305 1 u
mass of $^{26}_{14}$Si	25.992 330 u

So, does the mass of an atom equal the sum of the masses of its constituents? A ${}^4_2$He atom consists of two electrons, two protons, and two neutrons:

sum of masses of constituents	4.032 979 6 u
mass of ${}^4_2$He	4.002 603 2 u

No! The atom is less massive than the sum of its constituents!

Problem: Compare the masses of the following systems, each of which has the same constituents: (a) four electrons, four protons, and four neutrons (b) two ${}^4_2$He atoms, and (c) one ${}^8_4$Be atom.

four times mass of (electron plus proton plus neutron)	8.065 959 2 u
mass of ${}^8_4$Be	8.005 305 1 u
twice mass of ${}^4_2$He	8.005 206 4 u

Problem: The molecule acetylene, $H - C \equiv C - H$, consists of 14 electrons, 14 protons, and 12 neutrons. (Assuming that it's made from the most abundant isotopes of carbon and hydrogen, namely ${}^{12}_6$C and ${}^1_1$H.) The atom silicon-26 (${}^{26}_{14}$Si) has exactly the same constituents.

sum of masses of 14 electrons, 14 protons, and 12 neutrons	26.213 527 4 u
mass of acetylene molecule	26.015 650 0 u
mass of ${}^{26}_{14}$Si	25.992 330 u

5.11 Problems

1. *A trip to the grocery store.*

Jane Morowitz needs to pick up some butter from a grocery store located 1200 feet from her home. She travels there at a speed of $V = \frac{4}{5}c$. As she starts her journey, her wristwatch, her home's wall clock, and the grocery store's wall clock all read noon.

a. How much time (in the earth's frame) does it take her to reach the grocery store? (Give your result in nans.)

b. How much time ticks off on her wristwatch during this journey?

c. In Jane's frame, the grocery store moves towards her. How far must it travel to reach her?

d. Using your result for part (c) and the fact that

$$\text{distance} = \text{speed} \times \text{time},$$

find out how much time it takes for the grocery store to reach Jane. Compare to your result from part (b).

e. In Jane's frame, the grocery store's wall clock is moving, so it ticks slowly. How much time does it tick off during its journey to reach Jane?

f. When Jane reaches the grocery store, its wall clock will read the time you found in part (a). How can it read this time when it has ticked off only the amount of time you determined in part (e)?

2. *Muon lifetime.*

The lifetime of a stationary muon is 2.2 μs. A beam of muons is produced traveling at 83% of the speed of light. How far will the muons travel before they decay?

3. *Engine trouble.*

In the frame of a moving car, the driver coughs and at the same moment (by coincidence) a puff of smoke emerges from the tail pipe. In the earth's frame:

a. these two events are again simultaneous.
b. the puff of smoke emerges before the driver coughs.
c. the driver coughs before the puff of smoke emerges.
d. the driver never coughs.

Explain your answer in one sentence.

4. *Relativistic baking.*

A flat, horizontal tray of cookie dough speeds through a bakery at a substantial fraction of the speed of light. (The dough is square in the tray's reference frame, and thus rectangular in the baker's reference frame.) A baker stands ready with a circular cookie cutter, held horizontally, and as the tray of dough flashes by, she stamps out a cookie with lightning speed. (She drops and raises the cutter so quickly that nothing gets squashed or stuck in the cutter.) The resulting cookie will, in its own reference frame, be shaped like an oval rather than a circle. Is it longer in the direction of its motion or in the direction perpendicular to its motion? Justify your answer in both the tray's frame and the baker's frame. (Source: N.D. Mermin, *Space and Time in Special Relativity* (Waveland Press, Prospect Heights, Illinois, 1968) page 229.)

5. *Spy versus spy.*

James Bond and his old enemy Goldfinger each pilot a space ship: Bond is flying west, Goldfinger is flying east. The ships are identical except that Goldfinger has mounted a cannon at the tail of his ship, pointing perpendicular to the ship. Bond's ship is unarmed. Each ship is 150 feet long in its own rest frame. The two ships fly past each other with a relative speed of $V = \frac{4}{5}c$. In an attempt to do away with Bond once and for all, Goldfinger hatches the following diabolical plan. "I'll set my tail cannon on a timer to go off at noon, then I'll use my expert piloting skills to guide my ship so that my nose lines up with Bond's tail exactly at noon. My cannon will go off and automatically blast the nose off of Bond's ship!" Bond hears about this plan from a friend, the lithe and lovely Christine, planted in Goldfinger's crew. But it doesn't worry him: "My ship will be length contracted and Goldfinger's bullet will fly harmlessly in front of my bow," Bond claims. (Goldfinger is a man of limited intelligence who has never heard of length contraction.)

a. In Goldfinger's frame, how long is Bond's ship? Sketch the situation in Goldfinger's frame.

b. In Goldfinger's frame, how far in front of Bond's bow does the bullet fly?

Christine is more concerned about Bond's safety than Bond himself is. "But in *your* frame, Goldfinger's ship is contracted. Doesn't that mean that you'll be hit?"

c. In Bond's frame, which happens first, the bullet firing or the nose-tail lineup? Draw *two* sketches in Bond's frame that correspond to the one sketch from item (b).

d. Show that in Bond's frame, the timer at the rear of Goldfinger's ship is set 120 nans ahead of a clock in the nose of Goldfinger's ship.

e. While Goldfinger's clocks tick off 120 nans, how much time elapses in Bond's frame?

f. How far does Goldfinger's ship move in Bond's frame while Goldfinger's clocks tick off 120 nans?

g. In Bond's frame, the bullet again flies harmlessly in front of Bond's bow. How far in front of the bow does it pass?

h. How can we understand the difference between the results of items (b) and (g)? Suppose Bond's ship had been outfitted with a 180 foot flagpole protruding straight out from the bow. Goldfinger's bullet, although still non-fatal, would in this case snap the flagpole. Item (g) gives the length of the remnant in Bond's frame, while item (b) gives the length of the remnant in Goldfinger's frame. Are these two lengths related as you would expect?

6. *American graffiti: Length contraction revisited.*
Two commuters, Paul and Jennifer, stand on a railroad platform. Each has a felt-tip pen and a wristwatch. Although dressed conservatively for their workday business, each is at heart a secret anarchist and graffitist. They know that, at about 30 seconds before noon, the locomotive of a long express train will come roaring out of the west and rush by the platform at 3/5th the speed of light. The two agree that, exactly at noon, each will reach out with his/her felt-tip pen and make a mark on the passing train. Jennifer stands 100 feet to the east of Paul.

In the reference frame of the train:

a. Who marks first, Jennifer or Paul?

b. For how much time does the train have only one mark on it? (Remember, in the train's frame both watches tick slowly.)

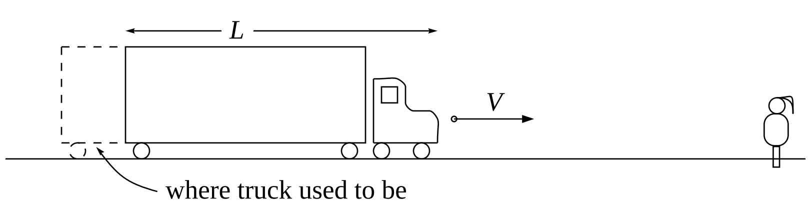
c. During the above time Paul has been traveling west (in the train's frame). How far does he travel?

d. How far apart are the two marks on the train? Compare to (100 feet)/$\sqrt{1-(V/c)^2}$.

(Clue: You will find this problem much easier if you familiarize yourself with this situation by making three sketches — one in the earth's frame corresponding to two in the train's frame.)

7. *Visual appearance of a moving truck.*
Sometimes students pick up the misimpression that the effects of special relativity are not real, but are appearance due to the finite speed of light.

For example, a truck of length L moves down the highway at speed V. Jessica, standing on the shoulder of the highway, watches the truck approach. Light from the nose of the truck takes some time to reach Jessica, but light from the tail of the truck takes even more time to reach her, because it has to travel further. Thus Jessica sees the nose of the truck as it was some time ago, but the tail of the truck as it was even more time ago, so the length she sees is longer than the true length of the truck.



This effect exists and is real, but it's not what we've been talking about in this course. When I say that a truck has length L_0 in its own reference frame and a shorter length L in the highway's reference frame, I mean that *really is* shorter, not just that it *appears* shorter due to the finite speed of light or some other effect. The visual-appearance *lengthening* effect cannot in any way explain relativistic length *contraction* — after all, it even goes the wrong direction!

Show that the visual-appearance length of the truck is

$$\frac{1}{1 - V/c} L.$$

8. *Time travel.*
Ivan's doctor breaks the sad news that Ivan has been diagnosed with an incurable disease and has just over two years to live. "Medical researchers are working on a cure right now," the doctor explains, "and it will be available in twelve years. But I'm afraid that this will be too late to help you." Instead of despairing, Ivan jumps into a fast rocket ship and travels at high speed. When he returns to Earth he has aged two years and is near death. But twelve years have elapsed on Earth, the medical research has been successful, and Ivan is cured. How fast did Ivan travel? (Ignore the brief periods of acceleration.)

9. *First to happen versus first to see.*
In the earth's frame: Two firecrackers, one red and one blue, are located 100 feet apart by the side of a highway. A rocket sled hurls down the highway at constant speed $V = \frac{4}{5}c$. Just as the rocket sled passes the red firecracker, both firecrackers explode. The rocket pilot sees first the red flash and then a bit later (because it takes some time for the light to reach her) she sees the blue flash.

In the rocket sled's frame: The two firecrackers are located 60 feet apart. First the blue (rear) firecracker explodes, then 80 nan later (80 nan $= L_0 V/c^2$) the red firecracker explodes (at the instant when it passes the rocket). Since the blue light travels 60 feet in a time of 60 nan, the pilot sees the blue light first and 20 nan later sees the red light.

Identify the argument's flaw. As usual, the key is to sketch three pictures: one in the earth's frame and two in the rocket sled's frame.

10. *Two fast women.*
Veronica speeds to the right past Ivan at speed $V = \frac{3}{5}c$. Veronica's sister Sybil speeds to the right past Ivan still faster, at speed $V = \frac{4}{5}c$ (in Ivan's frame). Ivan places two firecrackers by the side of the road: one is 15 feet to the right of the other. In Ivan's frame, the right-hand firecracker explodes 9 nan after the left-hand firecracker explodes. (Thus, in Ivan's frame these two events are separated by a distance $\Delta x = 15$ feet and by a time $\Delta t = 9$ nans.) What is the time interval between these two explosions in Veronica's frame? In Sybil's frame? Sketch these two events using five pictures: two in Ivan's frame, one in Veronica's, and two in Sybil's. *Moral:* Two events can occur in one sequence in one reference frame, simultaneous in another, and in the opposite sequence in a third.

11. *Two events.*
In the laboratory frame, two events are separated by $\Delta x = 14$ ft and $\Delta t = 6$ nan. Is there a frame in which these two events are simultaneous? If so, find the speed of that frame relative to the laboratory frame.

12. *Flushing out an error.*
Watch the music video "I Lost on Jeopardy" by "Weird Al" Yankovic, a parody of "[Our Love's in] Jeopardy" by the Greg Kihn Band:

http://www.youtube.com/watch?v=BvUZijEuNDQ

Find and correct the error in relativity.

13. *Interval.*
Show that the quantity $\Delta x^2 - (c\Delta t)^2$, sometimes called "interval", is the same in all reference frames, i.e. that

$$\Delta x^2 - (c\Delta t)^2 = \Delta x'^2 - (c\Delta t')^2.$$

Thus for the common sense Galilean transformation, Δt is the same in all reference frames, while interval is not. For the correct Lorentz transformation, the opposite holds.

14. *K^0 decay.*
When a K^0 meson at rest decays into a π^+ and a π^- meson, each escapes with a speed of approximately $0.73c$. A K^0 meson travels at speed $0.82c$ relative to the earth. When it decays, what is the greatest speed that one of the resulting π mesons can have in the earth's frame? What is the least speed? Compare your relativistic answers to the common sense answers. (These experiments have been performed, and they constitute one of the best test of the three principles — time dilation, length contraction, and the relativity of simultaneity — all wrapped into one.)

15. *Velocity addition formula.*
Using the Einstein velocity addition formula

$$v_b' = \frac{v_b - V}{1 - v_b V/c^2},$$

sketch v_b' as a function of v_b in the domain $V \le v_b < c$ for (a) $V = 1000$ mph; (b) $V = \frac{1}{2}c$; and (c) $V = \frac{3}{4}c$. For comparison, sketch also the non-relativistic formula $v_b' = v_b - V$.

16. *Relativistic energy and momentum, I.*
A particle of mass m is given so much energy that its total relativistic energy is equal to three times its rest energy. Find its resulting speed and momentum. How do these results change if the total energy is six times its rest energy?

17. *Relativistic energy and momentum, II.*
A particle of mass m has relativistic energy equal to γ times its rest energy (that is, $E = \gamma mc^2$). What is its speed? Its momentum?

18. *Relativistic energy: a new proposal.*
A friend tells you: "I have a new idea about relativistic energy. That old fogey Einstein got it all wrong! In fact, relativistic energy should be defined not as

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}} \qquad \text{but as} \qquad E = \frac{mc^2}{\sqrt{1-(v/c)^4}}."$$

Prove your friend wrong. (*Clue:* Examine the classical limit $v \ll c$ of this formula using the result that, when $|\epsilon| \ll 1$, $(1+\epsilon)^n \approx 1 + n\epsilon$.)

19. *Sticky particles.*

A putty ball of mass 5 kg is hurled at $v = \frac{12}{13}c$ towards a stationary putty ball of mass 2 kg. The two balls stick together. What is the mass and speed of the resulting lump of putty? (*Clue:* $\sqrt{1 - (\frac{12}{13})^2} = \frac{5}{13}$.)

20. *Sticky particles and the classical limit.*

A putty ball moving at speed v collides with an identical stationary putty ball. The two balls stick together.

a. In classical mechanics, what is the speed of the resulting composite?

b. In relativistic mechanics, what is the speed of the resulting composite?

c. Does your result in part (b) have the proper limit when $v \ll c$?

21. *Relativistic mass and thermal energy.*

A bottle of gas consists of N atoms, each of mass m and speed v, but moving in random directions so that the gas as a whole (as a "composite object") has velocity zero. Find the mass of the gas as a whole. In the case of $v \ll c$, interpret this mass as the sum of Nm plus the thermal energy divided by c^2.

22. *Particle creation.*

The Fermilab accelerator in Batavia, Illinois, gives a proton an energy of 300 GeV. That high-speed proton is then directed towards a stationary proton. The resulting collision can produce a new particle X through the reaction

$$\mathrm{p} + \mathrm{p} \longrightarrow \mathrm{p} + \mathrm{p} + \mathrm{X}$$

What is the largest possible rest mass M_{X} of a particle created in this way? The rest mass of a proton is about 0.939 GeV/c^2. (*Clue:* Use the conserved invariant $E^2 - (pc)^2$.)

Appendix A

A Sample Physics Problem

What are effective techniques for solving physics problems? How can solving problems help you understand physics? What does your teacher expect to find in your solutions? This document answers these questions through an example. It shows in action the principles described in the document "Solving Problems in Physics".

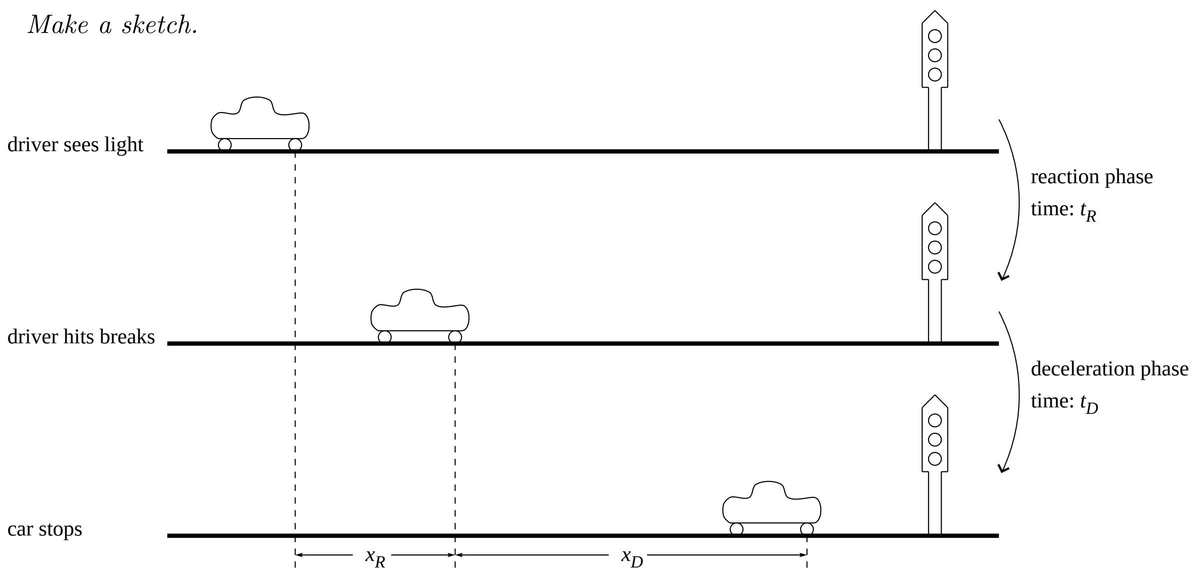
The problem

Here is our sample physics problem: (It is based on problem **2**–95 in HRW.)

> **2**–95. *Red light.* A car moves at constant speed when a traffic light ahead turns red. After a brief reaction time, the driver steps on the break petal and then the car slows with constant deceleration to a stop. The car requires 56.7 meters to stop from a speed of 80.5 km/hour, and 24.4 meters to stop from a speed of 48.3 km/hour. What is the reaction time of the driver and the rate of deceleration due to breaking? Discuss your numerical answer and the equations that lead up to it.

Finding a solution

Make a sketch.



(Note: There is an almost irresistible urge to draw a single sketch using superpositions to represent the three times, rather than three sketches representing the three times. I always find it clearer to show three separate sketches displaced vertically, as above. The clarity afforded by the expanded drawing translates into a more direct solution, which in the end saves time, ink, and paper space.)

What kind of problem is this? It's a hybrid of two problems: first (during the "reaction phase") it's a uniform speed problem, second (during the "deceleration phase") it's a uniform acceleration problem. (Although the acceleration a will be a negative number.) Let's refresh our understanding of these two types of motion through the chart below:

Relation:	uniform speed (v_0)	uniform acceleration (a)
v to t	$v = v_0$	$v = v_0 + at$
x to t	$x = x_0 + v_0 t$	$x = x_0 + v_0 t + \frac{1}{2}at^2$
v to x	$v = v_0$	$v^2 = v_0^2 + 2a(x - x_0)$

Where are we? Where do we want to go? This table outlines the situation, and at the same time defines the variables we'll use:

	reaction phase	deceleration phase
time required:	t_R	t_D
distance traveled:	x_R	x_D

We know $(x_R + x_D)$ and v_0 — we want to find t_R and a. It's clear that no one of the six equations above will do the job for us.

Strategy, first try. However, we can put together equations. For example, the two "relate x to t" equations can be applied to this situation as

$$x_R = v_0 t_R \qquad x_D = v_0 t_D + \tfrac{1}{2} a t_D^2.$$

And these two can be summed to produce

$$x_R + x_D = v_0(t_R + t_D) + \tfrac{1}{2} a t_D^2.$$

This equation looks useful: it relates things we know ($x_R + x_D$, v_0) to things we want to find (t_R, a), with only one extraneous quantity, namely t_D. (By "extraneous" I mean a quantity that we don't know and that we don't want to find.) You might think that we could write this equation once for the first case (56.7 meters from 80.5 km/hour) and once for the second case (24.4 meters from 48.3 km/hour), then eliminate t_D from the two resulting equations. This strategy fails because the deceleration time t_D will be different in the two different cases.

Strategy, second try. Since we don't know and don't care about the two different values of t_D for the two cases, let's not use the relation between x and t for the deceleration phase. Instead, we'll try the relation between v and x for this phase, namely

$$v^2 = v_0^2 + 2a(x - x_0)$$

which in our situation becomes

$$0 = v_0^2 + 2a x_D \quad \text{or} \quad x_D = -\frac{v_0^2}{2a}.$$

Combining this with our previous equation $x_R = v_0 t_R$ for the reaction phase gives

$$x_R + x_D = v_0 t_R - \frac{v_0^2}{2a}.$$

This looks like what we need! It has no extraneous quantities. Plugging in known numbers for the two cases will produce two equations, and solving the two equations simultaneously will find the two unknown quantities. Let's rush out and do it. . .

Check. No. Slow down. Before going through the labor of plugging in numbers, let's check this equation for reasonableness. The dimensions are correct. The result will be positive (remember that the acceleration a is negative). If the reaction time t_R increases, then the stopping distance $x_R + x_D$ increases. If the initial speed v_0 increases, then the stopping distance increases. If the acceleration a becomes more negative, then the stopping distance decreases. All of this makes sense. There's a brief moment of panic when we ask "What if $a = 0$? Then the resulting stopping distance is infinite!" But that's okay: if there's no deceleration, then the car never stops, so the stopping distance *should* be infinite. The result passes all our checks for reasonableness.

Plug in. Now we've arrived at a good place to put in numbers. Those numbers are:

	$x_R + x_D$	v_0
first case:	56.7 meters	80.5 km/hour = 22.4 m/sec
second case:	24.4 meters	48.3 km/hour = 13.4 m/sec

(Note the conversion to SI units.) So using

$$x_R + x_D = v_0 t_R - \frac{v_0^2}{2a}.$$

for the two cases we have (in meters and seconds):

$$\begin{aligned} 56.7 &= 22.4\ t_R - \frac{(22.4)^2}{2a} \\ 24.4 &= 13.4\ t_R - \frac{(13.4)^2}{2a}. \end{aligned}$$

Our plan is to eliminate the acceleration a first. To do this, divide the first equation by $(22.4)^2$ and the second by $(13.4)^2$, giving

$$\begin{aligned} 0.113 &= 0.0446\ t_R - \frac{1}{2a} \\ 0.136 &= 0.0746\ t_R - \frac{1}{2a}. \end{aligned}$$

Subtraction eliminates the acceleration:

$$0.023 = 0.0300\ t_R$$

or

$$t_R = 0.767 \text{ sec}.$$

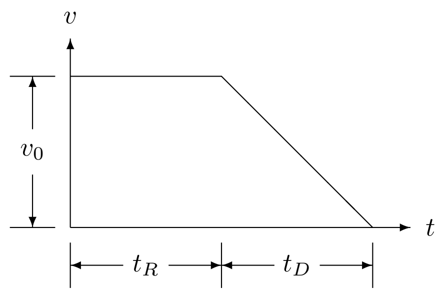
And plugging this into either of the two equations involving deceleration gives

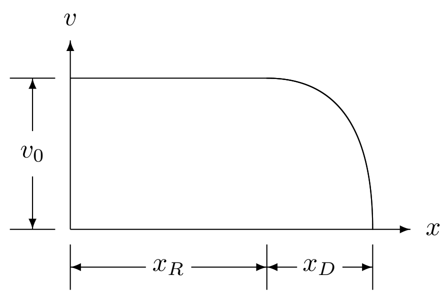
$$a = -6.34 \text{ m/sec}^2.$$

Both of these numbers seem reasonable to me: the reaction time is just under one second, and the deceleration is somewhat less than the acceleration of gravity $g = 9.8$ m/sec^2.

⟦True confession: The first time I did this problem I miscopied the stopping distance of 56.7 meters — I wrote 76.7 meters instead. I knew I had made an error when I got a negative reaction time.⟧

Graphs. Although the problem does not require this, it is informative to sketch graphs of v versus t and of v versus x.





Why do these graphs show t_R equal to t_D but x_R greater than x_D? Because the car is going faster during the reaction phase than during the deceleration phase, so if the times were equal then the reaction phase distance would be greater.

Writing up a solution

Your write-up doesn't need to describe all the blind alleys you went down in arriving at a solution. And it doesn't need to be flowing literate prose. But it does need to (1) show your reasoning, not just give the answer; (2) describe your thoughts in words and in figures as well as in equations; and (3) outline your checking. (When the problem statement says to "discuss", it usually means to outline your checks for reasonableness.)

The next page gives a full-credit answer to this problem. Things to note: (1) Define your symbols. (2) Choose mnemonic names for your symbols. (3) Don't present every arithmetic step (you're not in training to become a pocket calculator!), but do present every logical step. (4) Give numerical results with units and with the proper number of significant digits. (5) Include your name!

*Problem **2**–95: Red light.* *Dan Styer*

Initial speed v_0 maintained during reaction phase, followed by constant (negative) acceleration a.

	reaction phase	deceleration phase
time required:	t_R	t_D
distance traveled:	x_R	x_D

Known: $(x_R + x_D)$ and v_0
Desired: t_R and d.

Use $x = v_0 t$ during reaction phase: $x_R = v_0 t_R$

Use $v^2 = v_0^2 + 2a(x - x_0)$ during deceleration phase: $0 = v_0^2 + 2ax_D$ or $x_D = -\dfrac{v_0^2}{2a}$.

Add these two:

$$x_R + x_D = v_0 t_R - \frac{v_0^2}{2a}.$$

Checks:

Dimensions okay.
Stopping distance $x_R + x_D$ is positive.
Stopping distance increases with reaction time.
Stopping distance increases with initial speed.
Stopping distance decreases with greater deceleration (more negative a).
Correct result $x_R + x_D = \infty$ for special case $a = 0$.

Plug in numbers:

	$x_R + x_D$	v_0
first case:	56.7 meters	80.5 km/hour = 22.4 m/sec
second case:	24.4 meters	48.3 km/hour = 13.4 m/sec

So for the two cases:

$$\begin{aligned} 56.7\text{ m} &= (22.4\text{ m/sec})\, t_R - \frac{(22.4\text{ m/sec})^2}{2a} \\ 24.4\text{ m} &= (13.4\text{ m/sec})\, t_R - \frac{(13.4\text{ m/sec})^2}{2a}. \end{aligned}$$

Solving these two equations in two unknowns gives

$$t_R = 0.767\text{ sec}, \qquad a = -6.34\text{ m/sec}^2.$$

Check:
Numbers are reasonable: the reaction time just under one second, deceleration somewhat less than g.