Vector calculus: Geometrical definition of divergence and curl

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Derivative of a single-variable function

The derivative of function $f(x)$ at point x_0 is given by

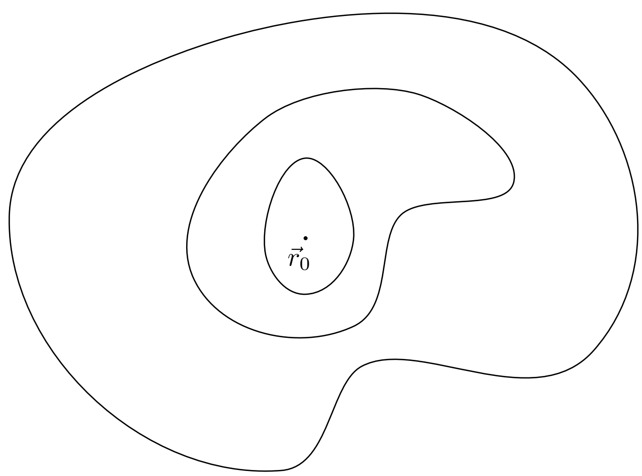
$$\lim_{L \to 0} \frac{f(x_0 + L/2) - f(x_0 - L/2)}{L}$$

that is, in involves the function values at the edges of an interval, divided by the magnitude of that interval.

If you're a mathematician, you'll want to prove that the limit exists, and that it is the same whether L approaches zero from above (through positive numbers) or from below (through negative numbers). Physicists usually just skip over such questions, interesting though they may be. Instead we just note that the result has the correct dimensions for a slope, and that it leads to the indeterminate form 0/0.

Divergence

So how should we define the derivative of a vector function $\vec{F}(\vec{r})$ at point $\vec{r}_0$? Here's one way. Consider a sequence of volumes $\mathcal{V}$ that enclose point $\vec{r}_0$, but that grow smaller and smaller.



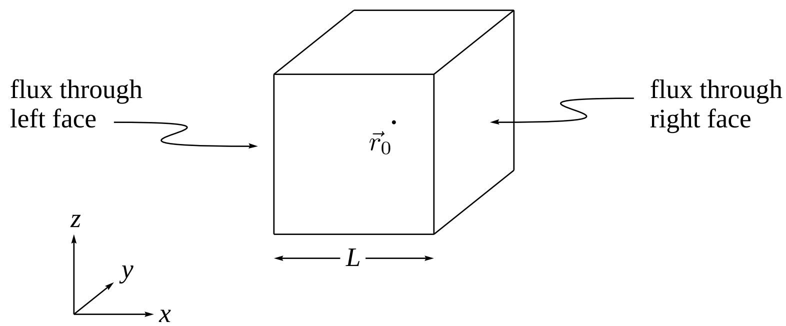
A sequence of volumes (shown in cross-section) homing in on point $\vec{r}_0$.

If the volume $\mathcal{V}$ is enclosed by surface $\mathcal{S}$, then

$$\lim \frac{\displaystyle\int_{\mathcal{S} \text{ of } \mathcal{V}} \vec{F}(\vec{r}) \cdot \hat{n}\, dA}{\text{volume of } \mathcal{V}} \tag{1}$$

fits the requirements for some sort of derivative: it involves the function values at the edge of the volume divided by the magnitude of that volume, it has the correct dimensions, and it leads to the indeterminate form 0/0. Mathematicians will want to prove that the limit exists, and that it gives the same result regardless of what sequence of volumes (cubes, spheres, hemispheres, cats, etc.) is used to close in on $\vec{r}_0$. But we'll skip over such general questions and ask:

What is the result if the sequence of volumes consists of cubes centered on $\vec{r}_0$?



It's clear from the definition of flux that for a small cube

$$\begin{aligned} \text{flux through right face} &\approx F_x(\text{evaluated at center of right face})L^2 \\ &= F_x(x_0 + L/2, y_0, z_0)L^2 \end{aligned}$$

and that this approximation grows better and better as L grows smaller and smaller. Similarly

$$\begin{aligned} \text{flux through left face} &\approx -F_x(\text{evaluated at center of left face})L^2 \\ &= -F_x(x_0 - L/2, y_0, z_0)L^2. \end{aligned}$$

Thus

$$\begin{aligned} \text{flux through right plus left faces} &\approx [F_x(x_0 + L/2, y_0, z_0) - F_x(x_0 - L/2, y_0, z_0)]L^2 \\ &= \left[\frac{F_x(x_0 + L/2, y_0, z_0) - F_x(x_0 - L/2, y_0, z_0)}{L}\right] L^3 \\ &\to \left[\frac{\partial F_x}{\partial x}(x_0, y_0, z_0)\right] L^3 \end{aligned}$$

where the symbol $\to$ means "in the limit as $L \to 0$".

Parallel reasoning shows that the flux through the back plus front faces is

$$\left[\frac{\partial F_y}{\partial y}(\vec{r}_0)\right] L^3$$

while the flux through the top plus bottom faces is

$$\left[\frac{\partial F_z}{\partial z}(\vec{r}_0)\right] L^3.$$

Finally, the limit presented in definition (1) results in

$$\frac{\partial F_x}{\partial x}(\vec{r}_0) + \frac{\partial F_y}{\partial y}(\vec{r}_0) + \frac{\partial F_z}{\partial z}(\vec{r}_0). \tag{2}$$

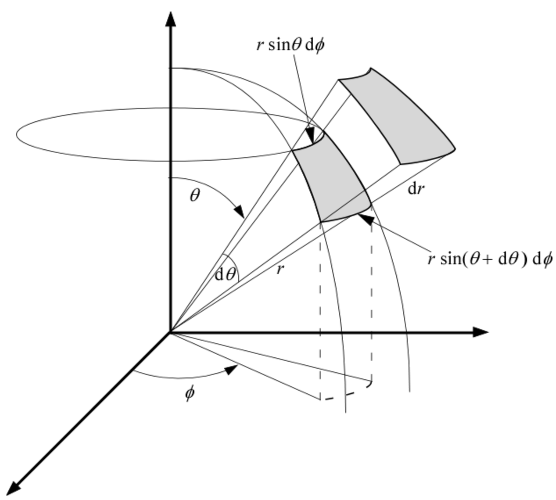
Because this derivative is the "flux per volume at a point" we call it the "divergence at a point".

Some people like to begin with equation (2) and call this the definition of divergence. Then they have a difficult time proving the *divergence theorem* (or Gauss's theorem), namely that if volume $\mathcal{V}$ is enclosed by surface $\mathcal{S}$, then

$$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{F}(\vec{r})\, d^3r = \int_{\mathcal{S}\text{ of }\mathcal{V}} \vec{F}(\vec{r}) \cdot \hat{n}\, dA.$$

I prefer to begin with the geometrical definition (1), and derive expression (2) for the divergence in Cartesian coordinates. In this approach, the divergence theorem just pops right out of the definition.

You could do the "flux through a shrinking volume" argument for shapes other than cubes. If you do it for the shape below



you will find the expression for divergence in spherical coordinates. If you do it for other shapes you will find the expression for divergence in cylindrical coordinates, or prolate spheroidal coordinates, or confocal paraboloidal coordinates, or any other kind of coordinates. The idea of "flux per volume at a point" is the same for all coordinate systems – the shape of the shrinking volume is different for different coordinate systems.

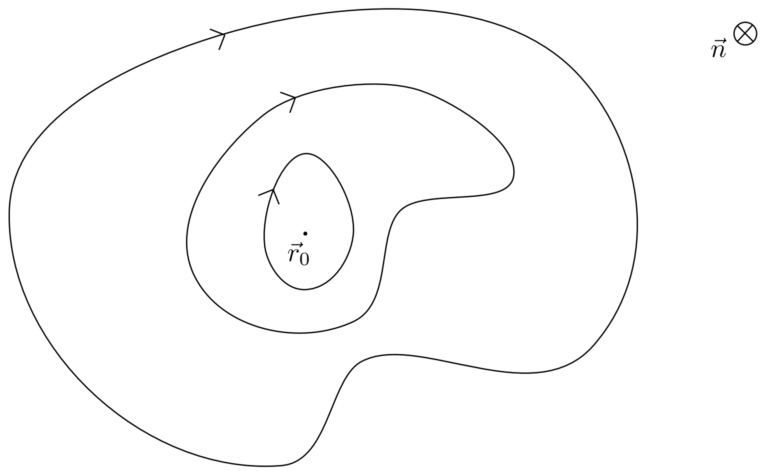
Summary: You know that the total mass M of an object can be found by integrating the mass density $\rho(\vec{r})$ over the volume of the object:

$$M = \int_{\mathcal{V}} \rho(\vec{r})\, d^3r.$$

The divergence plays the role of "flux density" rather than mass density.

Curl

Expression (1) is not the only possible derivative of a vector function. Consider a sequence of loops $\mathcal{L}$ (all within the plane perpendicular to a unit vector $\hat{n}$) that enclose point $\vec{r}_0$, but that grow smaller and smaller.



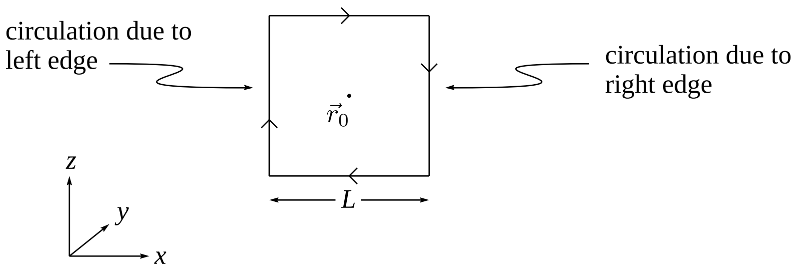
A sequence of loops (all within the plane perpendicular to $\hat{n}$) homing in on point $\vec{r}_0$.
(Direction of loop given through right-hand rule.)

The line integral of $\vec{F}(\vec{r})$ along loop $\mathcal{L}$ is called the "circulation of $\vec{F}(\vec{r})$ along $\mathcal{L}$." If the loop $\mathcal{L}$ embraces a surface $\mathcal{S}$, then

$$\lim \frac{\int_{\mathcal{L} \text{ of } \mathcal{S}} \vec{F}(\vec{r}) \cdot d\vec{\ell}}{\text{area of } \mathcal{S}} \tag{3}$$

also fits the requirements for some sort of derivative: it involves the function values at the edge of the surface divided by the magnitude of that surface, it has the correct dimensions, and it leads to the indeterminate form 0/0. Mathematicians will want to prove that the limit exists, and that it gives the same result regardless of what sequence of shapes (squares, circles, squirrels, etc.) is used to close in on $\vec{r}_0$. But we'll skip over such general questions and ask:

What is the result if the sequence of loops consists of squares in the x-z plane, centered on $\vec{r}_0$?



It's clear from the definition of circulation that for a small square

$$\begin{aligned}\text{circulation due to right edge} &\approx -F_z(\text{evaluated at center of right edge})L \\ &= -F_z(x_0 + L/2, y_0, z_0)L\end{aligned}$$

and that this approximation grows better and better as L grows smaller and smaller. Similarly

$$\begin{aligned}\text{circulation due to left edge} &\approx F_z(\text{evaluated at center of left edge})L \\ &= F_z(x_0 - L/2, y_0, z_0)L.\end{aligned}$$

Thus

$$\begin{aligned}\text{circulation due to right plus left edges} &\approx -[F_z(x_0 + L/2, y_0, z_0) - F_z(x_0 - L/2, y_0, z_0)]L \\ &= -\left[\frac{F_z(x_0 + L/2, y_0, z_0) - F_z(x_0 - L/2, y_0, z_0)}{L}\right]L^2 \\ &\to -\left[\frac{\partial F_z}{\partial x}(x_0, y_0, z_0)\right]L^2\end{aligned}$$

where the symbol $\to$ means "in the limit as $L \to 0$". Parallel reasoning shows that the circulation due to the top plus bottom faces is

$$\left[\frac{\partial F_x}{\partial z}(\vec{r}_0)\right]L^2.$$

Finally, the limit presented in definition (3) results in

$$\frac{\partial F_x}{\partial z}(\vec{r}_0) - \frac{\partial F_z}{\partial x}(\vec{r}_0). \tag{4}$$

Because this derivative is the "circulation in the plane perpendicular to y per area at a point" we call it the "y-component of curl at a point".

Parallel considerations for planes perpendicular to x and to z result in the traditional expression for the curl in Cartesian coordinates, namely

$$\vec{\nabla} \times \vec{F}(\vec{r}_0) \doteq \left[\frac{\partial F_z}{\partial y}(\vec{r}_0) - \frac{\partial F_y}{\partial z}(\vec{r}_0), \frac{\partial F_x}{\partial z}(\vec{r}_0) - \frac{\partial F_z}{\partial x}(\vec{r}_0), \frac{\partial F_y}{\partial x}(\vec{r}_0) - \frac{\partial F_x}{\partial y}(\vec{r}_0)\right]. \tag{5}$$

Some people like to begin with equation (5) and call it the definition of curl. Then they have a difficult time proving the *circulation theorem* (or Stokes's theorem): If the surface $\mathcal{S}$ is bounded by loop $\mathcal{L}$, then

$$\int_{\mathcal{S}} (\vec{\nabla} \times \vec{F}(\vec{r})) \cdot \hat{n} \, dA = \int_{\mathcal{L} \text{ of } \mathcal{S}} \vec{F}(\vec{r}) \cdot d\vec{\ell}. \tag{6}$$

I prefer to begin with the geometrical definition

$$(\vec{\nabla} \times \vec{F}(\vec{r}_0)) \cdot \hat{n} = \lim \frac{\displaystyle\int_{\mathcal{L} \text{ of } \mathcal{S}} \vec{F}(\vec{r}) \cdot d\vec{\ell}}{\text{area of } \mathcal{S}}, \tag{7}$$

and derive expression (5) for the Cartesian coordinates of the curl. (Just as the gradient $\vec{\nabla} f(\vec{r}_0)$ points in the direction of fastest increase of $f(\vec{r}_0)$, and has magnitude equal to the slope in that direction, so $\vec{\nabla} \times \vec{F}(\vec{r}_0)$ points in the direction of largest circulation/area, and has magnitude equal to the circulation/area in that direction.) In this approach, the circulation theorem just pops right out of the definition.

Acknowledgment: In my multivariate calculus course, I learned the "Cartesian coordinate" definitions of divergence and curl, and these definitions left a bad taste in my mouth. Why were divergence and curl – particularly curl – defined through such bazaar combinations of derivatives? Math is supposed to be coordinate-independent: Why were Cartesian coordinates so special? A one-variable derivative has geometrical significance as a slope – what was the geometrical significance of divergence and curl? A related question – why do divergence and curl have such strange names?

I learned the much-more-satisfactory geometric approach to vector derivatives – the one outlined in this document – from my physics professor Mark Heald. When I asked him how he had learned it, he told me it was the approach used by his teacher, Carl Howe, when he took undergraduate electricity and magnetism at Oberlin College.