The geometrical significance of the Laplacian

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Abstract

The Laplacian operator can be defined, not only as a differential operator, but also through its averaging properties. Such a definition lends geometric significance to the operator: a large Laplacian at a point reflects a "nonconformist" (i.e., different from average) character for the function there. This point of view is used to motivate the wave equation for a drumhead.

Keywords:

Laplacian, geometry, electrostatics, relaxation, intuition, nonconformist, nonconformity, analogy, wave equation, waves on a drumhead, waves on a membrane

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I. INTRODUCTION

The Laplacian operator is encountered throughout physics: It appears in the wave equation, Schrödinger's equation, the equations for irrotational fluid flow, the diffusion equation, Poisson's equation, and of course Laplace's equation. How can this one mathematical tool play a role in such diverse phenomena as sound and light, quantum mechanics and classical mechanics, heat and concentration and electrostatics?

Let me put the question in a different way. The Laplacian of a scalar function $f(x, y, z)$ is typically defined through

$$\vec{\nabla}^2 f(x,y,z) = \vec{\nabla} \cdot \vec{\nabla} f(x,y,z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{1}$$

What makes this particular combination of partial derivatives so special? Why do we so often encounter this combination and so rarely encounter, say, the combination

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \quad ? \tag{2}$$

The answer is that Laplacian is not *just* a jumble of symbols, but is also a reflection of geometry.[1,2] I will show that the Laplacian of a function at point $\vec{r}_0$ can be *defined* as

$$\vec{\nabla}^2 f(\vec{r}_0) = \lim_{R\to 0} \left\{ \frac{6}{R^2} \left[\langle f \rangle_{\text{shell}} - f(\vec{r}_0) \right] \right\}, \tag{3}$$

where $\langle f \rangle_{\text{shell}}$ means the average value of $f(\vec{r})$ on the surface of a sphere of radius R centered on $\vec{r}_0$. This definition is an exact parallel to the geometrically insightful definition of the second derivative (which is the Laplacian in one dimension), namely

$$\frac{d^2}{dx^2} f(x_0) = \lim_{R\to 0} \left\{ \frac{2}{R^2} \left[\langle f \rangle_{\text{shell}} - f(x_0) \right] \right\}, \tag{4}$$

where here $\langle f \rangle_{\text{shell}} = [f(x_0 - R) + f(x_0 + R)]/2$. Equations like the two above are examples of what will be called an "averaging property".

If, at a given point $\vec{r}_0$, the function is the same as the average over surrounding points, then the Laplacian vanishes. On the other hand if the function is far from that average then the Laplacian is far from zero. A person is called a "nonconformist" if s/he differs from the average of the people immediately surrounding her/him — by analogy we say that the Laplacian measures the "nonconformity" of a function at a point.

This analogy — like all analogies — is imperfect. (If an analogy *were* perfect, then it would be just as difficult to understand the analogous situation as the original situation.)

When applied to people, the word “conformist” suggests not only that the person is similar to the average of the surrounding people, but also that those surrounding people are similar to each other. This is *not* necessary for the “conformist” (i.e. zero-Laplacian) function. In the sense intended here, a person would be a “conformist” in height if s/he were surrounded by people both taller and shorter than her/him; or by many persons a little shorter and one person considerably taller.

Defining the Laplacian through an averaging property has several advantages.[3–6] For example, the averaging property can be used as the basis for the relaxation method.[7–9] The averaging definition extends immediately to the Laplacian of a vector function, whereas the “divergence of gradient” definition leads any thoughtful undergraduate to puzzle over “what’s the gradient of a vector function?”. But most importantly, the averaging property provides geometric insight into the character of the operator.

The body of this paper demonstrates the equivalence of these two definitions for the Laplacian operator in three dimensions, and then — as an application — motivates the wave equation for waves on a drumhead using the “conformist” analogy. Appendix A is historical and quotes James Clerk Maxwell’s treatment of the Laplacian, which is similar to ours (if more telegraphic!). Appendix B concerns the Laplacian operator in three dimesions: it assumes the traditional differential definition and proves an averaging property (valid for a sphere of any radius, not just for $R \to 0$); Appendix C does the converse. The remaining Appendices D and E consider the Laplacian in arbitrary positive integral dimension. For those who don’t care to slog through higher-dimension swamps, I present here the result: For dimensionality d, the generalization of definitions (3) and (4) is

$$\vec{\nabla}^2 f(\vec{r}_0) = \lim_{R\to 0}\left\{\frac{2d}{R^2}\left[\langle f\rangle_{\text{shell}} - f(\vec{r}_0)\right]\right\}. \tag{5}$$

II. FROM AVERAGING PROPERTY TO DIFFERENTIAL EXPRESSION

Can one really define the Laplacian (in three dimensions) through equation (3) rather than through the familiar equation (1)?

Yes. The average value of function $f(\vec{r})$ over the surface (shell) of a sphere of radius R

centered on point $\vec{r}_0$ is defined as

$$\langle f \rangle_{\text{shell}} = \frac{\int_{\text{shell}} f(\vec{r})\, d^2r}{\int_{\text{shell}} d^2r} = \frac{\int_{\text{shell}} f(\vec{r})\, d^2r}{4\pi R^2}, \tag{6}$$

so equation (3) states that

$$\begin{aligned} \vec{\nabla}^2 f(\vec{r}_0) &= \lim_{R\to 0} \left\{ \frac{6}{R^2} \left[\frac{\int_{\text{shell}} f(\vec{r})\, d^2r}{4\pi R^2} - f(\vec{r}_0) \right] \right\} \\ &= \lim_{R\to 0} \left\{ \frac{6}{R^2} \left[\frac{\int_{\text{shell}} [f(\vec{r}) - f(\vec{r}_0)]\, d^2r}{4\pi R^2} \right] \right\} \\ &= \lim_{R\to 0} \left\{ \frac{6}{4\pi R^4} \int_{\text{shell}} [f(\vec{r}) - f(\vec{r}_0)]\, d^2r \right\}. \end{aligned} \tag{7}$$

For notational convenience, move the origin of the coordinate system to point $\vec{r}_0$, which will thus be called $\vec{0}$. The Taylor series expansion of the integrand of equation (7) is (all partial derivatives evaluated at $\vec{0}$)

$$\begin{aligned} f(\vec{r}) - f(\vec{0}) = &\frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y + \frac{\partial f}{\partial z} z \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2} x^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2} y^2 + \frac{1}{2}\frac{\partial^2 f}{\partial z^2} z^2 \\ &+ \frac{\partial^2 f}{\partial x \partial y} xy + \frac{\partial^2 f}{\partial x \partial z} xz + \frac{\partial^2 f}{\partial y \partial z} yz + \cdots. \end{aligned} \tag{8}$$

By symmetry,

$$\int_{\text{shell}} x\, d^2r = 0 \quad \text{and} \quad \int_{\text{shell}} xy\, d^2r = 0, \qquad \text{etc.,} \tag{9}$$

but

$$\begin{aligned} &\int_{\text{shell}} x^2\, d^2r = \int_{\text{shell}} y^2\, d^2r = \int_{\text{shell}} z^2\, d^2r \\ &= \tfrac{1}{3} \int_{\text{shell}} (x^2 + y^2 + z^2)\, d^2r = \tfrac{1}{3} \int_{\text{shell}} R^2\, d^2r = \tfrac{4}{3}\pi R^4. \end{aligned} \tag{10}$$

Thus equation (7) becomes

$$\begin{aligned} \vec{\nabla}^2 f(\vec{0}) &= \lim_{R\to 0} \left\{ \frac{6}{4\pi R^4} \left(\frac{1}{2}\right) \left(\tfrac{4}{3}\pi R^4\right) \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \cdots \right] \right\} \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \end{aligned} \tag{11}$$

a more familiar expression for the Laplacian!

III. WAVES ON A DRUMHEAD

A membrane stretches taut over the open end of a drum. The membrane is struck so that it deviates from tautness: this deviation is $f(x,y)$.

If, at some given point, the deviation is "conformist" — identical to the average deviation around it — then there is no net force on this patch of membrane so it doesn't accelerate.

On the other hand, if the deviation is "nonconformist" — say it is higher then the average membrane height around it (i.e. the Laplacian of f at this point is negative) — then there is a net force downward, so the patch accelerates downward.

Similarly, if the patch is lower than the average surrounding it (i.e. the Laplacian of f at this point is positive), then there is a net force upward, so the patch accelerates upward.

Ignoring longitudinal forces, nonlinearities, etc., we have that

$$\vec{\nabla}^2 f(x,y,t) = \kappa \frac{\partial^2 f(x,y,t)}{\partial t^2} \tag{12}$$

where κ is a positive constant. This is as far as the "nonconformist" analogy by itself can take us. But the dimensions of κ are clearly (time/length)^2, so it makes sense to express this constant through a new constant with the dimensions of velocity $v_p \equiv \kappa^{-1/2}$, giving the wave equation in form

$$\vec{\nabla}^2 f(x,y,t) = \frac{1}{v_p^2} \frac{\partial^2 f(x,y,t)}{\partial t^2}. \tag{13}$$

The constant v_p will, of course, turn out to be the phase velocity of waves on the drumhead.

This argument should be considered a motivation, not a derivation, and does not replace those rigorous arguments[10] which establish the connection between surface tension, mass density, and wave velocity. On the other hand it is far easier and far more insightful than those derivations.

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My teacher and friend Mark A. Heald insisted that we students should understand the geometrical significance as well as the coordinate expressions for vector operators like gradient, divergence, and curl. He taught us never to take a vector identity for granted, but

always to look into what that identity said about geometry. This wise advice has helped me several times in my career and has given birth to this paper. (In addition, Professor Heald critiqued a draft of this essay.) In grateful acknowledgment I dedicate this essay to him.

Appendix A: Maxwell's approach to the Laplacian

In his 1881 *Treatise on Electricity and Magnetism*,[11] James Clerk Maxwell (who defined the Laplacian as the negative of the Laplacian that we define today) wrote that

> One of the most remarkable properties of the operator ∇ is that when repeated it becomes
>
> $$\nabla^2 = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right),$$
>
> an operator occurring in all parts of Physics, which we may refer to as Laplace's Operator. . . .
>
> If, with any point P as centre, we draw a small sphere whose radius is r, then if q_0 is the value of q at the centre, and $\bar{q}$ the mean value of q for all points within the sphere,
>
> $$q_0 - \bar{q} = \tfrac{1}{10} r^2 \nabla^2 q\,;$$
>
> so that the value at the centre exceeds or falls short of the mean value according as $\nabla^2 q$ is positive or negative.
>
> I propose therefore to call $\nabla^2 q$ the *concentration* of q at the point P, because it indicates the excess of the value of q at that point over its mean value in the neighbourhood of the point.

One can easily integrate our "average over the surface of a sphere" result (3) to derive Maxwell's "average over the interior of a sphere" result above.

Appendix B: From differential definition to general averaging property

In this appendix we give the Laplacian operator its standard differential definition

$$\vec{\nabla}^2 f(\vec{r}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{B1}$$

and prove this general averaging property: that if $\langle f\rangle_{\text{shell}}$ is the average value of $f(\vec{r})$ on the surface of a sphere of radius R centered on $\vec{r}_0$, then

$$\langle f\rangle_{\text{shell}} = f(\vec{r}_0) + \frac{1}{4\pi}\int_{\text{ball}} \left(\vec{\nabla}^2 f(\vec{r})\right)\left(\frac{1}{|\vec{r}-\vec{r}_0|} - \frac{1}{R}\right) d^3r, \tag{B2}$$

where the integral ranges over the interior of the sphere.

Proof part I: Alternate expressions for the general averaging property. Begin by defining (suggestively for anyone who has studied electrostatics)

$$\vec{\nabla}^2 f(\vec{r}) \equiv -\rho(\vec{r}). \tag{B3}$$

With this definition the general averaging property to be proven becomes

$$\begin{aligned}
\langle f\rangle_{\text{shell}} &= f(\vec{r}_0) - \frac{1}{4\pi}\int_{\text{ball}} \rho(\vec{r})\left(\frac{1}{|\vec{r}-\vec{r}_0|} - \frac{1}{R}\right) d^3r \\
&= f(\vec{r}_0) - \frac{1}{4\pi}\int_{\text{ball}} \frac{\rho(\vec{r})}{|\vec{r}-\vec{r}_0|}\, d^3r + \frac{1}{4\pi R}\int_{\text{ball}} \rho(\vec{r})\, d^3r && \text{(B4)} \\
&= f(\vec{r}_0) - f^{(\text{inside})}(\vec{r}_0) + \frac{Q^{(\text{inside})}}{4\pi R} && \text{(B5)}
\end{aligned}$$

where, in the last step, we (a) define

$$Q^{(\text{inside})} = \int_{\text{ball}} \rho(\vec{r})\, d^3r \tag{B6}$$

and (b) recognize the Coulomb's law expression for electrostatic potential (which can be derived from equation B1) and define $f^{(\text{inside})}(\vec{r}_0)$ as the solution of Poisson's equation that would result if $\rho(\vec{r})$ vanished outside the sphere (that is, the electrostatic potential due to "inside charges"). With a parallel definition for $f^{(\text{outside})}(\vec{r}_0)$, the averaging property becomes

$$\langle f\rangle_{\text{shell}} = f^{(\text{outside})}(\vec{r}_0) + \frac{Q^{(\text{inside})}}{4\pi R}. \tag{B7}$$

Proof strategy. We will first prove the result in form (B7) for

$$\rho(\vec{r}) = q\delta(\vec{r}-\vec{r}_s), \tag{B8}$$

("a point charge located at $\vec{r}_s$"). The full theorem follows immediately through superposition.

Proof part II: For a point source charge.[12] The Poisson equation

$$\vec{\nabla}^2 f(\vec{r}) = -q\delta(\vec{r}-\vec{r}_s) \tag{B9}$$

has the well-known (and easily verified) solution

$$f(\vec{r}) = \frac{1}{4\pi}\frac{q}{|\vec{r}-\vec{r}_s|}. \tag{B10}$$

For the "point charge located at $\vec{r}_s$" situation, the result to be proven is thus

$$\langle f\rangle_{\text{shell}} = f^{(\text{outside})}(\vec{r}_0) = \frac{q}{4\pi}\frac{1}{|\vec{r}_0-\vec{r}_s|} \qquad \text{for } R < |\vec{r}_0-\vec{r}_s| \tag{B11}$$

$$\langle f\rangle_{\text{shell}} = \frac{q}{4\pi R} \qquad \text{for } R > |\vec{r}_0-\vec{r}_s|, \tag{B12}$$

or, using the definition of average (6) plus the solution (B10),

$$\frac{1}{4\pi R^2}\int_{\text{shell}} \frac{1}{4\pi}\frac{q}{|\vec{r}-\vec{r}_s|}\, d^2r = \frac{q}{4\pi}\frac{1}{|\vec{r}_0-\vec{r}_s|} \qquad \text{for } R < |\vec{r}_0-\vec{r}_s| \tag{B13}$$

$$\frac{1}{4\pi R^2}\int_{\text{shell}} \frac{1}{4\pi}\frac{q}{|\vec{r}-\vec{r}_s|}\, d^2r = \frac{q}{4\pi R} \qquad \text{for } R > |\vec{r}_0-\vec{r}_s|. \tag{B14}$$

To test this possibility we could evaluate these integrals directly, at enormous cost in blood and toil. But a trick based on Gauss's Law permits their evaluation indirectly, and simply.

Consider a completely different problem: There is no longer a charge at $\vec{r}_s$, instead charge q is uniformly spread over the shell in question, generating a surface charge density of $q/4\pi R^2$. What is the potential at point $\vec{r}_s$? The expression for that potential is

$$\int_{\text{shell}} \frac{1}{4\pi}\frac{q/4\pi R^2}{|\vec{r}-\vec{r}_s|}\, d^2r, \tag{B15}$$

and according to the shell theorem, this potential evaluates to

$$\frac{q}{4\pi}\frac{1}{|\vec{r}_0-\vec{r}_s|} \qquad \text{for } R < |\vec{r}_0-\vec{r}_s| \tag{B16}$$

$$\frac{q}{4\pi R} \qquad \text{for } R > |\vec{r}_0-\vec{r}_s|. \tag{B17}$$

Thus the theorem is proved. Q.E.D.

Appendix C: From general averaging definition to differential property

In this appendix we define the Laplacian operator through the general averaging property

$$\langle f\rangle_{\text{shell}} = f(\vec{r}_0) + \frac{1}{4\pi}\int_{\text{ball}} \left(\vec{\nabla}^2 f(\vec{r})\right)\left(\frac{1}{|\vec{r}-\vec{r}_0|} - \frac{1}{R}\right) d^3r, \tag{C1}$$

for any sphere of radius R centered on any point $\vec{r}_0$, and prove that the Laplacian operator is then realized through the differential expression

$$\vec{\nabla}^2 f(\vec{r}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{C2}$$

Proof: For notational convenience, move the origin of the coordinate system to point $\vec{r}_0$, which will thus be called $\vec{0}$. The averaging property in form (B4) is thus

$$\langle f\rangle_{\text{shell}} = f(\vec{0}) - \frac{1}{4\pi}\int_{\text{ball}} \frac{\rho(\vec{r})}{|\vec{r}|}\, d^3r + \frac{1}{4\pi R}\int_{\text{ball}} \rho(\vec{r})\, d^3r. \tag{C3}$$

Since this form holds for a sphere of any size, it holds for one so small that the variation of $\rho(\vec{r})$ across the sphere is negligible. For such a sphere

$$\langle f\rangle_{\text{shell}} - f(\vec{0}) = -\frac{1}{4\pi}\rho\int_{\text{ball}} \frac{1}{|\vec{r}|}\, d^3r + \frac{1}{4\pi R}\rho\int_{\text{ball}} d^3r = -\tfrac{1}{6}R^2\rho. \tag{C4}$$

Solving for $\rho = -\vec{\nabla}^2 f$ gives

$$\vec{\nabla}^2 f(\vec{r}_0) = \lim_{R\to 0}\left\{\frac{6}{R^2}\left[\langle f\rangle_{\text{shell}} - f(\vec{r}_0)\right]\right\}, \tag{C5}$$

which is equation (3), the starting point of section II. The arguments of section II then generate the result

$$\vec{\nabla}^2 f(\vec{r}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{C6}$$

Q.E.D.

Appendix D: Result in dimensionality $d \neq 2$

Theorem: The Laplacian operator may be defined, for positive integral dimensionality $d \neq 2$, through any of the three equivalent expressions

$$\vec{\nabla}^2 f(\vec{r}) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_d^2}, \tag{D1}$$

$$\langle f\rangle_{\text{shell}} = f(\vec{r}_0) + \frac{\Gamma(d/2)}{2\pi^{d/2}(d-2)}\int_{\text{ball}} \left(\vec{\nabla}^2 f(\vec{r})\right)\left(\frac{1}{|\vec{r}-\vec{r}_0|^{d-2}} - \frac{1}{R^{d-2}}\right) d^dr, \tag{D2}$$

$$\vec{\nabla}^2 f(\vec{r}_0) = \lim_{R\to 0}\left\{\frac{2d}{R^2}\left[\langle f\rangle_{\text{shell}} - f(\vec{r}_0)\right]\right\}. \tag{D3}$$

Note. Remarkably, this result holds even for $d = 1$.

Proof. The proofs are a straightforward, almost robotic, generalization of the proofs in appendices B and C. Here I explicate the only three points likely to cause difficulty.

Point 1: Solution to Poisson's equation. At equation (B10), I took it for granted that you knew the solution to Poission's equation for a point source in three dimensions. Here, we generalize to dimension $d \neq 2$ by searching for a solution to

$$\vec{\nabla}^2 f(\vec{r}) = -q\delta(x_1)\delta(x_2)\delta(x_3)\cdots\delta(x_d). \tag{D4}$$

First, establish the notation

$$\vec{r} \equiv (x_1, x_2, x_3, \ldots, x_d) \quad \text{and} \quad r \equiv [x_1^2 + x_2^2 + x_3^2 + \cdots + x_d^2]^{1/2}. \tag{D5}$$

Then attempt a trial solution of the form

$$F(\vec{r}) = \frac{q}{Ar^\alpha} = \frac{q}{A[x_1^2 + x_2^2 + x_3^2 + \cdots + x_d^2]^{\alpha/2}}. \tag{D6}$$

This trial solution has

$$\frac{\partial F}{\partial x_i} = \frac{q}{A}\left(-\frac{\alpha}{r^{\alpha+1}}\right)\frac{\partial r}{\partial x_i} = -\frac{q}{A}\left(\frac{\alpha}{r^{\alpha+1}}\right)\frac{x_i}{r} = -\frac{\alpha q}{A}\frac{x_i}{r^{\alpha+2}}$$

whence

$$\vec{\nabla} F(\vec{r}) = -\frac{\alpha q}{A}\frac{\vec{r}}{r^{\alpha+2}}. \tag{D7}$$

And it has

$$\frac{\partial^2 F}{\partial x_i^2} = -\frac{\alpha q}{A}\left[\frac{1}{r^{\alpha+2}} - (\alpha+2)\frac{x_i^2}{r^{\alpha+4}}\right] = -\frac{\alpha q}{Ar^{\alpha+2}}\left[1 - (\alpha+2)\frac{x_i^2}{r^2}\right]$$

whence

$$\vec{\nabla}^2 F(\vec{r}) = -\frac{\alpha q}{Ar^{\alpha+2}}(d - \alpha - 2). \tag{D8}$$

The trial solution satisfies $\vec{\nabla}^2 F(\vec{r}) = 0$ for $\vec{r} \neq \vec{0}$ when

$$\alpha = d - 2. \tag{D9}$$

To check whether the trial solution holds at $\vec{r} = \vec{0}$, form a hypersphere of radius R centered on the origin. Integrate both sides of equation (D4) over the volume of this hypersphere. If $F(\vec{r})$ is to be a solution, it must satisfy

$$\int_{\text{ball}} \vec{\nabla}^2 F(\vec{r})\, dx_1 dx_2 dx_3 \cdots dx_d = -q. \tag{D10}$$

However, for any function $f(\vec{r})$,

$$\int_{\text{ball}} \vec{\nabla}^2 f(\vec{r})\, d^d r = \int_{\text{ball}} \vec{\nabla} \cdot \vec{\nabla} f(\vec{r})\, d^d r = \int_{\text{shell}} \vec{\nabla} f(\vec{r}) \cdot \hat{n}\, d^{d-1} r. \tag{D11}$$

On the shell, $\hat{n} = \vec{r}/r$, so

$$\vec{\nabla} F(\vec{r}) \cdot \hat{n} = -\frac{\alpha q}{A}\frac{\vec{r}}{r^{\alpha+2}} \cdot \frac{\vec{r}}{r} = -\frac{\alpha q}{A}\frac{1}{r^{\alpha+1}} \tag{D12}$$

whence

$$\int_{\text{shell}} \vec{\nabla} F(\vec{r}) \cdot \hat{n}\, d^{d-1}r = -\frac{\alpha q}{A} \frac{1}{R^{\alpha+1}} \times \text{ (surface area of hypersphere).} \tag{D13}$$

The surface area of a hypersphere with radius R in dimensionality $d \geq 1$ is

$$\text{surface area} \equiv S_d R^{d-1} \tag{D14}$$

where, as it happens,[13]

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{D15}$$

but in fact we will rarely need this evaluation of S_d. Instead, we note that

$$\int_{\text{shell}} \vec{\nabla} F(\vec{r}) \cdot \hat{n}\, d^{d-1}r = -q\frac{\alpha S_d}{A} R^{d-\alpha-2}. \tag{D16}$$

Equation (D10) demands that, if $F(\vec{r})$ is to be a solution, then this integral must equal $-q$ for all values of R, whence again we derive the requirement $\alpha = d-2$. In addition,

$$A = (d-2)S_d.$$

In conclusion, the solution to equation (D4), for $d \neq 2$, is

$$f(\vec{r}) = \frac{1}{(d-2)S_d} \frac{q}{r^{d-2}}. \tag{D17}$$

Point 2: Gauss's Law in d dimensions. Gauss's law and the shell theorem is invoked in section II just below equation (B14). Can these legitimately be used in the d-dimensional case? Yes, because Gauss's law holds for a force field that (a) exhibits superposition and (b) is proportional to $\hat{r}/r^{d-1}$ (or, what is the same thing, a potential proportional to $1/r^{d-2}$).

Point 3: Small sphere limit. Applying the averaging property (D2) to a small sphere (analogous to equation C4) results in

$$\begin{aligned}
\langle f \rangle_{\text{shell}} - f(\vec{0}) &= \frac{1}{(d-2)S_d} \int_{\text{ball}} \left(\vec{\nabla}^2 f(\vec{r})\right) \left(\frac{1}{|\vec{r}|^{d-2}} - \frac{1}{R^{d-2}}\right) d^d r \\
&= \frac{1}{(d-2)S_d} \rho \left[-\int_{\text{ball}} \frac{1}{|\vec{r}|^{d-2}}\, d^d r + \frac{1}{R^{d-2}} \int_{\text{ball}} d^d r \right] \\
&= \frac{1}{(d-2)S_d} \rho \left[-\int_0^R \frac{S_d r^{d-1}}{r^{d-2}}\, dr + \frac{1}{R^{d-2}} \int_0^R S_d r^{d-1}\, dr \right] \\
&= \frac{1}{(d-2)} \rho \left[-\int_0^R r\, dr + \frac{1}{R^{d-2}} \int_0^R r^{d-1}\, dr \right] \\
&= -\rho \frac{R^2}{2d}.
\end{aligned}$$

Note in particular that S_d cancels numerator and denominator in this calculation. This was the point of the remark, immediately below equation (D15), that "we will rarely need this evaluation of S_d".

Appendix E: Result in dimensionality $d = 2$

Theorem: The Laplacian operator may be defined, for dimensionality $d = 2$, through any of the three equivalent expressions

$$\vec{\nabla}^2 f(\vec{r}) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}, \tag{E1}$$

$$\langle f \rangle_{\text{circle}} = f(\vec{r}_0) - \frac{1}{2\pi} \int_{\text{disk}} \left(\vec{\nabla}^2 f(\vec{r}) \right) \ln \frac{|\vec{r} - \vec{r}_0|}{R} \, d^2 r, \tag{E2}$$

$$\vec{\nabla}^2 f(\vec{r}_0) = \lim_{R \to 0} \left\{ \frac{4}{R^2} \left[\langle f \rangle_{\text{circle}} - f(\vec{r}_0) \right] \right\}. \tag{E3}$$

Proof. Once again, the proofs follow the $d \neq 2$ case closely, and I note only two points.

Point 1: Solution to Poisson's equation. The solution to

$$\vec{\nabla}^2 f(x, y) = -q \delta(x) \delta(y) \tag{E4}$$

is readily checked to be

$$f(x, y) = -\frac{q}{2\pi} \ln(r/r^*), \tag{E5}$$

where $r = \sqrt{x^2 + y^2}$ and r^* is any arbitrary constant with the dimensions of length.

Point 2: Small disk limit. Apply result (E2) to a disk centered on the origin $\vec{0}$, using the familiar notation $\vec{\nabla}^2 f(\vec{r}) = -\rho(\vec{r})$:

$$\langle f \rangle_{\text{circle}} = f(\vec{0}) + \frac{1}{2\pi} \int_{\text{disk}} \rho(\vec{r}) \ln(r/R) \, d^2 r. \tag{E6}$$

If the disk is small enough that $\rho(\vec{r})$ may be considered constant throughout, then

$$\begin{aligned}
\langle f \rangle_{\text{circle}} &= f(\vec{0}) + \frac{1}{2\pi} \rho \int_0^R \ln(r/R) \, 2\pi r \, dr \\
&= f(\vec{0}) + \rho \int_0^R r \ln(r/R) \, dr \\
&= f(\vec{0}) + \rho R^2 \int_0^1 x \ln x \, dx \\
&= f(\vec{0}) + \rho R^2 \left(-\tfrac{1}{4} \right)
\end{aligned}$$

whence

$$\vec{\nabla}^2 f(\vec{r}_0) = \lim_{R \to 0} \left\{ \frac{4}{R^2} \left[\langle f \rangle_{\text{circle}} - f(\vec{r}_0) \right] \right\}. \tag{E7}$$

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[1] Richard P. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964) volume II, pages 12-12 to 12-13.

[2] The geometrical approach to vector operators has been particularly advocated by H.M. Schey, *Div, Grad, Curl, and All That: An Informal Text on Vector Calculus* (W.W. Norton & Company, New York, 1973). But Schey does not provide any geometrical insight into the Laplacian.

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[11] James Clerk Maxwell, *A Treatise on Electricity and Magnetism*, second edition (Clarendon Press, Oxford, UK, 1881) volume I, pages 29–30.

[12] This argument is modified from one presented in Paul Lorrain and Dale Corson, *Electromagnetic Fields and Waves*, second edition (W.H. Freeman, San Francisco, 1970) pages 49–51.

[13] Eric W. Weisstein, "Hypersphere", from MathWorld — A Wolfram Web Resource. ⟨http://mathworld.wolfram.com/Hypersphere.html⟩.