Model Solution for "Space: The Final Frontier"

(a.) *Time to collision.* Use Kepler's second law in Kepler's form, namely

$$\text{area swept out in time } t \ = (\ell/2m_E)t,$$

where ℓ is the angular momentum about the force center (in this case, the gas cloud). That angular momentum is clearly $\ell = m_E v_0 R_0$. The *Enterprise* will strike the gas cloud after it has swept out area equal to half a circle, that is $\frac{1}{2}\pi(R_0/2)^2$. Thus

$$\text{time to collision} \ = \frac{\text{area}}{(\ell/2m_E)} = \frac{\pi R_0^2/8}{v_0 R_0/2} = \frac{\pi}{4}\frac{R_0}{v_0}.$$

That is, the time to collision is slightly less than it would be for going straight into the gas cloud while maintaining the current velocity.

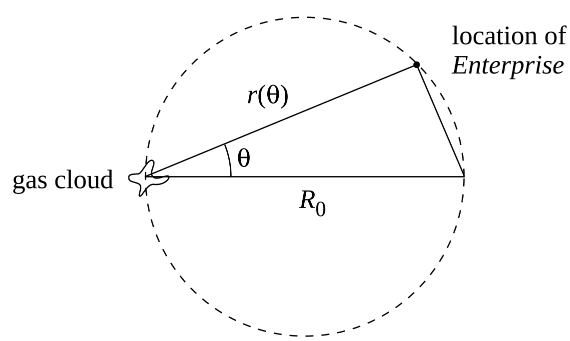
(b.) *Force law.* The differential equation for the orbit $r(\theta)$ is

$$\frac{\ell^2}{2m_E r^4(\theta)}\left(\frac{dr(\theta)}{d\theta}\right)^2 = E - \left[U(r) + \frac{\ell^2}{2m_E r^2}\right],$$

where E is the total energy. Usually this formula is used starting with a known potential energy function $U(r)$ to find the orbit $r(\theta)$, but in this case we use it in reverse, and write

$$U(r) = E - \frac{\ell^2}{2m_E r^2(\theta)}\left[1 + \left(\frac{1}{r(\theta)}\frac{dr(\theta)}{d\theta}\right)^2\right].$$

To find $r(\theta)$ just look at this figure:



It's clear that $r(\theta) = R_0 \cos\theta$.

Thus

$$\frac{dr(\theta)}{d\theta} = -R_0 \sin\theta$$

and

$$
\begin{aligned}
U(r) &= E - \frac{\ell^2}{2m_E r^2(\theta)}\left[1+\left(\frac{1}{r(\theta)}\frac{dr(\theta)}{d\theta}\right)^2\right] \\
&= E - \frac{\ell^2}{2m_E r^2}\left[1+\left(\frac{-R_0\sin\theta}{R_0\cos\theta}\right)^2\right] \\
&= E - \frac{\ell^2}{2m_E r^2}\left[\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}\right] \\
&= E - \frac{\ell^2}{2m_E r^2}\left[\frac{1}{\cos^2\theta}\right] \\
&= E - \frac{\ell^2}{2m_E r^2}\left[\frac{R_0^2}{r^2}\right].
\end{aligned}
$$

To have the conventional limit $U(r) \to 0$ as $r \to \infty$, we must set the potential energy zero so that E vanishes. (That is, $U(R_0) = -\frac{1}{2}m_E v_0^2$.)

In conclusion, the potential energy function is

$$
U(r) = -\frac{\ell^2 R_0^2/2m_E}{r^4} = -\frac{(\frac{1}{2}m_E v_0^2)R_0^4}{r^4}.
$$

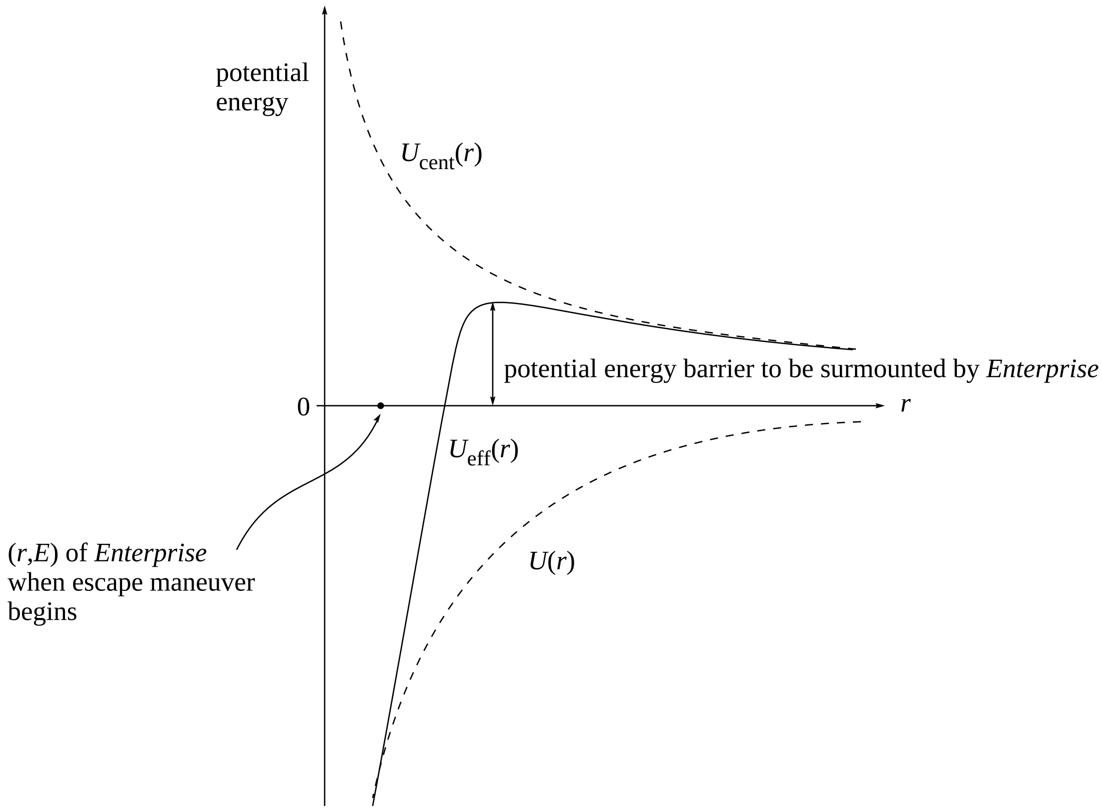
(c.) *Energy required for escape.* Because the engines are fired "directly away from the gas cloud", the force provided by the exhaust on the *Enterprise* delivers no torque to the *Enterprise*. Thus the angular momentum $\ell = m_E v_0 R_0$ stays constant while the engines fire, whence the centrifugal potential

$$
U_{\text{cent}}(r) = +\frac{\ell^2}{2m_E r^2} = (\tfrac{1}{2}m_E v_0)\frac{R_0^2}{r^2}
$$

is not altered during the escape maneuver. The effective potential energy diagram is given on the next page.

The potential energy barrier is located at point $\hat{r}$ where $U_{\text{eff}}(r)$ is maximized. That is, the point where

$$
\begin{aligned}
0 &= \frac{dU_{\text{eff}}(r)}{dr} \\
0 &= \frac{dU(r)}{dr} + \frac{dU_{\text{cent}}(r)}{dr} \\
0 &= -(\tfrac{1}{2}m_E v_0^2)\frac{d}{dr}\left(\frac{R_0^4}{r^4}\right) + (\tfrac{1}{2}m_E v_0^2)\frac{d}{dr}\left(\frac{R_0^2}{r^2}\right) \\
0 &= -R_0^4\left(-4\frac{1}{\hat{r}^5}\right) + R_0^2\left(-2\frac{1}{\hat{r}^3}\right) \\
2R_0^2\left(\frac{1}{\hat{r}^2}\right) &= 1 \\
\hat{r}^2 &= 2R_0^2 \\
\hat{r} &= \sqrt{2}R_0
\end{aligned}
$$



At this maximizing point, the effective potential energy is

$$U_{\text{eff}}(\hat{r}) = U(\hat{r}) + U_{\text{cent}}(\hat{r}) = -(\tfrac{1}{2}m_E v_0^2)\frac{R_0^4}{\hat{r}^4} + (\tfrac{1}{2}m_E v_0)\frac{R_0^2}{\hat{r}^2} = \tfrac{1}{4}(\tfrac{1}{2}m_E v_0^2).$$

This is the amount of kinetic energy the engines must generate before the *Enterprise* is freed (assuming engines fired "directly away from the gas cloud").