Variational principle for the harmonic oscillator

$$\psi(x) = \frac{A}{x^2+b^2} \qquad \text{let} \qquad \tilde{x} \equiv \frac{x}{b}$$

Normalization: ⟦Integral from Dwight 120.2.⟧

$$\begin{aligned}
1 &= \int_{-\infty}^{+\infty} \psi^* \psi \, dx = \int_{-\infty}^{+\infty} \frac{|A|^2}{(x^2+b^2)^2} \, dx = |A|^2 \int_{-\infty}^{+\infty} \frac{1}{(b^2\tilde{x}^2+b^2)^2} b \, d\tilde{x} = \frac{|A|^2}{b^3} \int_{-\infty}^{+\infty} \frac{d\tilde{x}}{(\tilde{x}^2+1)^2} \\
&= \frac{|A|^2}{b^3} \left[\frac{\tilde{x}}{2(\tilde{x}^2+1)} + \frac{1}{2} \arctan \tilde{x} \right]_{-\infty}^{+\infty} = \frac{|A|^2}{b^3} \left[\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right) \right] = \frac{|A|^2}{b^3} \frac{\pi}{2}.
\end{aligned}$$

Kinetic energy:

$$\begin{aligned}
\langle \mathrm{KE} \rangle &= -\frac{\hbar^2}{2m} |A|^2 \int_{-\infty}^{+\infty} \frac{1}{x^2+b^2} \frac{d^2}{dx^2} \left(\frac{1}{x^2+b^2} \right) dx = -\frac{\hbar^2}{2m} |A|^2 \int_{-\infty}^{+\infty} \frac{1}{b^2\tilde{x}^2+b^2} \frac{1}{b^2} \frac{d^2}{d\tilde{x}^2} \left(\frac{1}{b^2\tilde{x}^2+b^2} \right) b \, d\tilde{x} \\
&= -\frac{\hbar^2}{2m} \frac{|A|^2}{b^5} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\tilde{x}^2+1} \frac{d^2}{d\tilde{x}^2} \left(\frac{1}{\tilde{x}^2+1} \right) d\tilde{x}}_{\equiv -C_K} = \frac{\hbar^2}{2m} \frac{2}{\pi b^2} C_K
\end{aligned}$$

[This makes sense: a narrow wavefunction (small b) will have large curvature and hence large kinetic energy.]

Potential energy:

$$\begin{aligned}
\langle \mathrm{PE} \rangle &= \frac{m\omega^2}{2} \langle x^2 \rangle = \frac{m\omega^2}{2} |A|^2 \int_{-\infty}^{+\infty} \frac{x^2}{(x^2+b^2)^2} \, dx = \frac{m\omega^2}{2} |A|^2 \int_{-\infty}^{+\infty} \frac{b^2\tilde{x}^2}{(b^2\tilde{x}^2+b^2)^2} b \, d\tilde{x} \\
&= \frac{m\omega^2}{2} \frac{|A|^2}{b} \underbrace{\int_{-\infty}^{+\infty} \frac{\tilde{x}^2}{(\tilde{x}^2+1)^2} \, d\tilde{x}}_{\equiv C_P} = \frac{m\omega^2}{2} \frac{2b^2}{\pi} C_P
\end{aligned}$$

[This makes sense: a wide wavefunction (large b) will extend into the region of high potential energy.]

Total energy:

$$\langle H \rangle = \frac{\hbar^2}{2m} \frac{2}{\pi b^2} C_K + \frac{m\omega^2}{2} \frac{2b^2}{\pi} C_P$$

To *minimize* $\langle H \rangle$, take the derivative with respect to b and set equal to zero:

$$\frac{\partial \langle H \rangle}{\partial b} = \frac{2}{\pi} \left[-\frac{\hbar^2}{2m} \frac{2}{b^3} C_K + \frac{m\omega^2}{2} 2b C_P \right] = 0 \quad \Longrightarrow \quad b^2 = \frac{\hbar}{m\omega} \sqrt{\frac{C_K}{C_P}}$$

This value of b^2 results in

$$\langle H \rangle_{\min} = \frac{2}{\pi} \hbar\omega \sqrt{C_K C_P}.$$

Evaluate the integrals: ⟦For the C_K integral, use parts followed by Dwight 122.4.⟧

$$\begin{aligned}
C_K &= -\int_{-\infty}^{+\infty} \frac{1}{\tilde{x}^2+1}\frac{d^2}{d\tilde{x}^2}\left(\frac{1}{\tilde{x}^2+1}\right) d\tilde{x} = \int_{-\infty}^{+\infty} \frac{d}{d\tilde{x}}\left(\frac{1}{\tilde{x}^2+1}\right)\frac{d}{d\tilde{x}}\left(\frac{1}{\tilde{x}^2+1}\right) d\tilde{x} \\
&= \int_{-\infty}^{+\infty} \left[\frac{d}{d\tilde{x}}\left(\frac{1}{\tilde{x}^2+1}\right)\right]^2 d\tilde{x} = \int_{-\infty}^{+\infty} \left[\frac{-2\tilde{x}}{(\tilde{x}^2+1)^2}\right]^2 d\tilde{x} = 4\int_{-\infty}^{+\infty} \frac{\tilde{x}^2}{(\tilde{x}^2+1)^4}\, d\tilde{x} \\
&= 4\left[\frac{1}{16}\arctan\tilde{x}\right]_{-\infty}^{+\infty} = \frac{1}{4}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{4} \\
C_P &= \int_{-\infty}^{+\infty} \frac{\tilde{x}^2}{(\tilde{x}^2+1)^2}\, d\tilde{x} = \left[\frac{1}{2}\arctan\tilde{x}\right]_{-\infty}^{+\infty} = \frac{\pi}{2}
\end{aligned}$$

So

$$\langle H\rangle_{\min} = \frac{1}{\sqrt{2}}\hbar\omega = 0.707\,\hbar\omega > \frac{1}{2}\hbar\omega.$$

Grading: 2 points for each of sections "Normalization", "Kinetic energy", "Potential energy", "Minimization", and "Evaluate".