Quantal recurrence in the infinite square well

a. Classical period:

$$E = \tfrac{1}{2}mv^2 \quad \text{so} \quad v = \sqrt{2E/m}$$

and

$$\text{distance} = \text{speed} \times \text{time},$$

so

$$\text{period} = \frac{\text{distance}}{\text{speed}} = \frac{2L}{\sqrt{2E/m}} = L\sqrt{\frac{2m}{E}}. \tag{1}$$

b. Quantal recurrence:

How does the initial wavefunction $\psi(x;0)$ change with time? Expanded the initial wavefunction into energy eigenfunctions $\eta_n(x)$:

$$\psi(x;0) = \sum_{n=1}^{\infty} c_n \eta_n(x). \tag{2}$$

This wavefunction evolves in time to

$$\psi(x;t) = \sum_{n=1}^{\infty} c_n e^{-iE_n t/\hbar} \eta_n(x), \tag{3}$$

where the eigenvalues are

$$E_n = \frac{\pi^2\hbar^2}{2ML^2} n^2 = E_1 n^2 \quad \text{for} \quad n = 1, 2, 3, \ldots\ . \tag{4}$$

The time-evolved wavefunction will equal the initial wavefunction whenever all of the phase factors $e^{-iE_n t/\hbar}$ are equal to one. That is, the revival occurs at a time T_{rev} where

$$\frac{E_n}{\hbar} T_{\text{rev}} = 2\pi \text{ (some integer)}$$

for all values of n. Using the eigenenergy result this becomes

$$\frac{E_1}{\hbar} T_{\text{rev}} n^2 = 2\pi \text{ (some integer)}$$

so the revival time is

$$T_{\text{rev}} = \frac{2\pi\hbar}{E_1} = \frac{h}{E_1} = \frac{4mL^2}{\pi\hbar}. \tag{5}$$

(Note that we solved this part knowing only the energy eigenvalues.)

c. What happens after one-half of this time has passed?

Evaluated at $t = T_{\text{rev}}/2$, equation (3) gives

$$\psi(x;T_{\text{rev}}/2) = \sum_{n=1}^{\infty} c_n e^{-iE_n T_{\text{rev}}/2\hbar} \eta_n(x). \tag{6}$$

But $T_{\text{rev}} = h/E_1$, so

$$\frac{E_n T_{\text{rev}}}{2\hbar} = \frac{E_1 T_{\text{rev}}}{2\hbar} n^2 = \pi n^2$$

and

$$\psi(x; T_{\rm rev}/2) = \sum_{n=1}^{\infty} c_n e^{-i\pi n^2} \eta_n(x).$$

Now

$$e^{-i\pi n^2} = (-1)^{n^2} = (-1)^n$$

so

$$\psi(x; T_{\rm rev}/2) = \sum_{n=1}^{\infty} c_n (-1)^n \eta_n(x). \tag{7}$$

But the energy eigenfunction $\eta_n(x)$ is even for n odd and odd for n even, so

$$(-1)^n \eta_n(x) = -\eta_n(-x)$$

whence

$$\psi(x; T_{\rm rev}/2) = -\psi(-x; 0). \tag{8}$$

That is: After half a revival time, the initial wavefunction is flipped from left to right and turned up-side down (that is, multiplied by the physically-irrelevant overall phase factor of -1). (Note that we solved this part knowing only the energy eigenvalues and the parity of the energy eigenfunctions.)

Grading:

2 points for part **a**.
2 points for equation (3).
2 more points for finishing off part **b**.
2 points for reaching equation (7).
2 more points for finishing off part **c**.