Quantal recurrence in the Coulomb problem

As in part (**b**) of the problem "Quantal recurrence in the infinite square well," we ask how the initial wavefunction $\psi(\vec{r};0)$ changes with time. In terms of the energy eigenfunctions $\eta_n(\vec{r})$,

$$\psi(\vec{r};0) = \sum_n c_n \eta_n(\vec{r}).$$

This wavefunction evolves in time to

$$\psi(\vec{r};t) = \sum_n c_n e^{-iE_n t/\hbar} \eta_n(\vec{r}). \tag{1}$$

The revival comes when all the relevant phase factors equal one. (By "relevant" I mean all the phase factors that enter into the superposition, that is, the phase factors for those eigenstates for which $c_n \neq 0$.) This revival occurs at a time T_{rev} where

$$\frac{E_n}{\hbar} T_{\text{rev}} = 2\pi \text{ (some integer)} \qquad \text{whenever } c_n \neq 0.$$

The eigenvalues are now of the form

$$-\frac{\text{Ry}}{n^2},$$

so the revival comes when

$$\frac{\text{Ry}}{n^2 h} T_{\text{rev}} = \text{ (some integer)} \qquad \text{whenever } c_n \neq 0.$$

In other words,

$$\left(\frac{\text{Ry}}{h} T_{\text{rev}} \right) \frac{1}{n^2} \text{ is an integer for any } n \text{ entering into the superposition.}$$

Thus

$$\frac{\text{Ry}}{h} T_{\text{rev}} \text{ is the least common multiple of } n_1^2, n_2^2, \cdots, n_r^2. \tag{2}$$

It's not hard to prove that

$$\text{LCM}(n_\alpha^2, n_\beta^2) = \text{LCM}^2(n_\alpha, n_\beta)$$

so, by induction

$$\frac{\text{Ry}}{h} T_{\text{rev}} = \text{LCM}^2(n_1, n_2, \ldots, n_r)$$

or

$$T_{\text{rev}} = \frac{h}{\text{Ry}} \text{LCM}^2(n_1, n_2, \ldots, n_r). \tag{3}$$

Grading:

2 points for equation (1)
6 points for equation (2)
2 points for polishing off to reach equation (3)