Mean separation

⟦1 point⟧ According to Griffiths equations (5.23) and (5.25), the mean square separations $\langle (x_A - x_B)^2 \rangle$ are

$$\begin{array}{rl} \text{for non-identical particles:} & \langle x^2 \rangle_n + \langle x^2 \rangle_m - 2\langle x \rangle_n \langle x \rangle_m \\ \text{for identical bosons/fermions:} & \text{the above} \mp 2\,|\langle m|x|n\rangle|^2 \end{array}$$

(Surprisingly, none of the integrals on the right involve integrands x_A or x_B, but simply x.)

⟦1 point⟧ Thus we need to perform three integrals. Well, not really. It's obvious that $\langle x \rangle_n = L/2$, for all values of n.

⟦3 points⟧ To find the mean of x^2 write

$$\begin{aligned} \langle x^2 \rangle_n &= \frac{2}{L}\int_0^L x^2 \sin^2\left(\frac{n\pi}{L}x\right)\,dx \qquad [\![\text{use substitution } u = (n\pi/L)x\ldots]\!] \\ &= \frac{2}{L}\left(\frac{L}{n\pi}\right)^3 \int_0^{n\pi} u^2 \sin^2 u\,du \qquad [\![\text{use Dwight equation 430.22}\ldots]\!] \\ &= \frac{2L^2}{n^3\pi^3}\left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{u}{4}\cos 2u\right]_0^{n\pi} \\ &= \frac{2L^2}{n^3\pi^3}\left[\frac{n^3\pi^3}{6} - \frac{n\pi}{4}\right] \\ &= L^2\left[\frac{1}{3} - \frac{1}{2n^2\pi^2}\right]. \end{aligned}$$

Thus

$$\langle x^2 \rangle_n + \langle x^2 \rangle_m - 2\langle x \rangle_n \langle x \rangle_m = L^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{m^2}\right)\right].$$

⟦4 points⟧ Meanwhile,

$$\begin{aligned} \langle m|x|n\rangle &= \frac{2}{L}\int_0^L x \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\,dx \qquad [\![\text{use substitution } \theta = (\pi/L)x\ldots]\!] \\ &= \frac{2}{L}\left(\frac{L}{\pi}\right)^2 \int_0^{\pi} \theta \sin m\theta \sin n\theta\,d\theta \\ &= \frac{2L}{\pi^2}\int_0^{\pi} \theta \tfrac{1}{2}[\cos(n-m)\theta - \cos(n+m)\theta]\,d\theta \\ &= \frac{L}{\pi^2}\left[\int_0^{\pi} \theta\cos(n-m)\theta\,d\theta - \int_0^{\pi}\theta\cos(n+m)\theta\,d\theta\right]. \end{aligned}$$

But for N an integer with $N \neq 0$,

$$\begin{aligned}
\int_0^\pi \theta \cos N\theta \, d\theta &= \frac{1}{N^2} \int_0^{N\pi} u \cos u \, du \\
&= \frac{1}{N^2} \left[\cos u + u \sin u \right]_0^{N\pi} \\
&= \frac{1}{N^2} \left[(-1)^N - 1 \right] \\
&= \frac{1}{N^2} \left\{ \begin{array}{rl} -2 & \text{for } N \text{ odd} \\ 0 & \text{for } N \text{ even} \end{array} \right.
\end{aligned}$$

So for $n - m$ even, $\langle m|x|n\rangle = 0$. But for $n - m$ odd

$$\begin{aligned}
\langle m|x|n\rangle &= \frac{L}{\pi^2}(-2) \left[\frac{1}{(n-m)^2} - \frac{1}{(n+m)^2} \right] \\
&= -\frac{8L}{\pi^2} \frac{nm}{(n^2 - m^2)^2}
\end{aligned}$$

so the pivotal term is

$$2|\langle m|x|n\rangle|^2 = \left\{ \begin{array}{cl} \dfrac{128\, L^2}{\pi^4} \dfrac{n^2 m^2}{(n^2 - m^2)^4} & \text{for } n,\ m \text{ of opposite parity} \\ 0 & \text{for } n,\ m \text{ of same parity} \end{array} \right.$$

⟦1 point⟧ In conclusion: For non-identical particles, or for bosons or fermions in the case that n and m are of the same parity, the root-mean-square separation is

$$\sqrt{\langle (x_A - x_B)^2 \rangle} = L \left[\frac{1}{6} - \frac{1}{2\pi^2} \left(\frac{1}{n^2} + \frac{1}{m^2} \right) \right]^{1/2}$$

while for bosons (minus sign) or fermions (plus sign) in the case that n and m are of opposite parity, the root-mean-square separation is

$$\sqrt{\langle (x_A - x_B)^2 \rangle} = L \left[\frac{1}{6} - \frac{1}{2\pi^2} \left(\frac{1}{n^2} + \frac{1}{m^2} \right) \mp \frac{128}{\pi^4} \frac{n^2 m^2}{(n^2 - m^2)^4} \right]^{1/2}.$$

There's a lot to explore in dissecting this result. Normally bosons huddle together whereas fermions spread apart. But when n and m are of the same parity, then bosons, fermions, and non-identical particles all have the same root-mean-square separation. Can you understand this any more deeply than just saying "it comes out of the math"? I can't.